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Statistics 100B **Instructor: Nicolas Christou**

Order statistics

Why order statistics?
We may be interested in
the fastest time in an automobile race,
the heaviest mouse among a group of mice fed on a certain diet,
the earliest time an electronic system fails,
the 1_{st} or n_{th} order statistics (could be estimates of parameters) etc.

Theory:
Let $X_1, X_2, \cdots, X_n$ denote independent continuous random variables with cdf $F(x)$ and pdf $f(x)$. We will denote the *ordered* random variables with $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$, where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ or $X_{(1)} = min(X_1, X_2, \cdots, X_n)$ and $X_{(n)} = max(X_1, X_2, \cdots, X_n)$. We call $X_{(1)}$ the *first* order statistic and $X_{(n)}$ the *nth* order statistic. Similarly, $X_{(j)}$ is the jth order statistic. We want to find the pdf of $X_{(1)}, X_{(n)}, X_{(j)}$, but also joint pdf functions that involve order statistics.

Useful results (see class notes for proofs):

a. Probability density function of the 1st order statistic.

$$g_{X_{(1)}}(x) = n\left[1 - F_X(x)\right]^{n-1} f_X(x)$$

b. Probability density function of the nth order statistic.

$$g_{X_{(n)}}(x) = n\left[F_X(x)\right]^{n-1} f_X(x)$$

c. Probability density function of the jth order statistic.

$$g_{X_{(j)}}(x) = \frac{n!}{(n-j)!(j-1)!}\left[F_X(x)\right]^{j-1}\left[1 - F_X(x)\right]^{n-j} f_X(x)$$

d. Joint probability density function of $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$.

$$g_{X_{(1)},X_{(2)},\ldots,X_{(n)}}(x_1, x_2, \ldots, x_n) = n! f_X(x_1) f_X(x_2) \ldots f_X(x_n)$$

e. Joint probability density function of $X_{(i)}, X_{(j)}$, with $1 \leq i < j \leq n$.

$$g_{X_{(i)},X_{(j)}}(u,v) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f_X(u) f_X(v) [F_X(u)]^{i-1} [F_X(v) - F_X(u)]^{j-1-i} [1 - F_X(v)]^{n-j}$$

Example 1:

Electronic components of a certain type have a length life (in hours) X, that follows the exponential distribution with probability density given by

$$f(x) = \frac{1}{100} e^{-\frac{1}{100}x}, \quad x > 0.$$

a. Suppose that 2 such components operate independently and in series in a certain system (that is, the system fails when either component fails). Find the density function for the length of life of the system.

b. Suppose that 2 such components operate independently and in parallel in a certain system (that is, the system does not fail until both components fail). Find the density function for the length of life of the system.

Example 2:

Let $X_1, X_2, \ldots, X_n$ i.i.d. $U(0, \theta)$. Find the pdf of $X_{(1)}, X_{(n)}, X_{(j)}$, and the joint pdf of $X_{(1)}, X_{(n)}$.