\[ \text{olf of } X: \quad P(X = x) = 1 - \frac{e^{-\lambda c}}{0!} = 1 - e^{-\lambda c} \]

Pdf of \( X: \quad f(x) = 2\pi x e^{-2\pi x} \]

Pdf of \( Y = \pi X^2 \)

Let \( W = \pi X^2 \)

\[ F_W(w) = P(W \leq w) = P(\pi X^2 \leq w) = P(X^2 \leq \frac{w}{\pi}) \]

\[ f_W(w) = \frac{1}{\pi} \frac{1}{2} w^{-\frac{1}{2}} e^{-\pi \frac{w}{2\pi}} = \frac{1}{\pi} \frac{1}{2} w^{-\frac{1}{2}} \] \( e^{-\pi \frac{w}{2\pi}} \]

\[ w \sim \exp(1) \]

\[ \text{Q is Poisson} \]
\[ \Pr(y \leq y) = 1 - \Pr(\alpha = 0 \text{ in ring}) = 2(\pi y^2 - \pi x^2) \]

with \[ \alpha = -2(\pi y^2 - \pi x^2) \]

\[ = 1 - \int_0^\infty e^{-x} \, dx = 1 - e^{-x} \]

pdf of \( y \):
\[ f(y) = 2\pi y \, e^{-\alpha(y^2 - x^2)} \]

pdf of \( \eta \, e^\alpha (y^2 - x^2) \):
\[ \text{let } z = \eta \, e^\alpha (y^2 - x^2) \]
\[ F_z(z) = \Pr(z \leq z) = \Pr \left( \eta \, e^\alpha (y^2 - x^2) \leq z \right) \]
\[ = \Pr \left( y^2 \leq \frac{z}{\eta} \right) = \Pr \left( y \leq \sqrt{\frac{z + \eta x^2}{\eta}} \right) \]
\[ = e^{-\ln \left( \sqrt{\frac{z + \eta x^2}{\eta}} \right)} = e^{-\ln (\sqrt{\frac{z + \eta x^2}{\eta}})} = e^{-\frac{z + \eta x^2}{\eta}} \]
\[ \therefore z \sim \exp(1) \]
c. Suppose now we randomly select \( m \) points in this forest. Find the distribution of \( 2\lambda^2 \sum_{i=1}^{m} X_i^2 \) and the distribution of \( 2\lambda^2 \sum_{i=1}^{m} (Y_i^2 - X_i^2) \).

Since \( \lambda X_i^2 \sim \exp(1) \) and \( \lambda (Y_i^2 - X_i^2) \sim \exp(1) \)

It follows that

\[
2\lambda \sum_{i=1}^{m} X_i^2 \sim \chi^2_m
\]

\[
2\lambda \sum_{i=1}^{m} (Y_i^2 - X_i^2) \sim \chi^2_m
\]

d. Let \( s = \lambda \sum_{i=1}^{m} X_i^2 \) and \( t = \lambda \sum_{i=1}^{m} (Y_i^2 - X_i^2) \). If \( s \) and \( t \) are independent show that \( \frac{\sum_{i=1}^{m} X_i^2}{\sum_{i=1}^{m} Y_i^2} \sim \text{beta}(m, m) \).

\[
S \sim \Gamma(m, 1), \quad T \sim \Gamma(m, 1)
\]

Let \( U = \frac{S}{S + T} \) and \( V = S + T \). \( S \) and \( T \) are independent. \( f_{S,T}(s,t) = \frac{s^{m-1} t^{m-1} e^{-s}}{\Gamma(m) \Gamma(m)} \)

\[
f_{UV}(u,v) = \int_{s=0}^{1} f_{S,T}(s=g_1(u,v), t=g_2(u,v)) \, ds
\]

\[
= \left( \frac{u}{\Gamma(m)} \right)^{m-1} \left( \frac{v(1-u)}{\Gamma(m)} \right)^{m-1} \frac{v}{\Gamma(2m)} \exp(-uv)
\]

\[
= \frac{v^{m-1} e^{-(uv + v(1-u))}}{\Gamma(m) \Gamma(m) \Gamma(2m)} \frac{v}{\Gamma(2m)}
\]

\[
\lambda (1 - V) \sim \Gamma(2m, 1)
\]

\[
U \sim \text{beta}(m, m)
\]