Quiz 10 solutions

Exercise 1

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma)$ μ IS UNKNOWN

$H_0: \sigma^2 = \sigma_0^2$

$H_a: \sigma^2 > \sigma_0^2$ MLE OF σ^2: $\hat{\sigma}_1^2 = \frac{\sum (X_i - \bar{X})^2}{n}$

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{(2\pi\sigma_0^2)^{-n/2} e^{-\frac{1}{2\sigma_0^2}\sum (X_i - \bar{X})^2}}{(2\pi\hat{\sigma}_1^2)^{-n/2} e^{-\frac{1}{2\hat{\sigma}_1^2}\sum (X_i - \bar{X})^2}} < K$$

$$\left(\frac{\hat{\sigma}_1^2}{\sigma_0^2}\right)^{n/2} e^{-\frac{1}{2\sigma_0^2}\sum (X_i - \bar{X})^2 + \frac{1}{2\hat{\sigma}_1^2}\sum (X_i - \bar{X})^2} < K$$

$$\left[\frac{\sum (X_i - \bar{X})^2}{n\sigma_0^2}\right]^{n/2} e^{-\frac{1}{2\sigma_0^2}\sum (X_i - \bar{X})^2 + \frac{n}{2}} < K$$

BUT $S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1}$

So

$$\left(\frac{(n-1)S^2}{\sigma_0^2}\right)^{n/2} \left(\frac{1}{n}\right)^{n/2} e^{-\frac{(n-1)S^2}{2\sigma_0^2}} \cdot e^{\frac{n}{2}} < K$$

$$\left(\frac{(n-1)S^2}{\sigma_0^2}\right)^{n/2} e^{-\frac{(n-1)S^2}{2\sigma_0^2}} < \frac{K}{e^{n/2}} n^{n/2}$$

WHICH IS A FUNCTION OF $\frac{(n-1)S^2}{\sigma_0^2}$

Exercise 2

Part (a).

1. Formulation of the two hypotheses:

$$H_0: \quad \mu = 130$$
$$H_a: \quad \mu < 130$$

2. We compute the test statistic z:

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{128.6 - 130}{\frac{3}{\sqrt{40}}} \Rightarrow z = -2.95$$

3. We find the rejection region. Here we use significance level $\alpha = 0.05$, therefore the rejection region will be when $z < -1.645$.

4. Conclusion: Since $z = -2.95 < -1.645$ we reject H_0. Therefore based on the evidence the mean output voltage is less than 130 volts.

Part (b). Yes, it is possible that the mean output voltage is still 130 volts. We may have commited a type I error.

Part(c). $1 - \beta = P(\bar{x} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} | \mu = \mu_a) = P(z < \frac{\mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_a}{\frac{\sigma}{\sqrt{n}}}) = P(z < \frac{130 - 1.645 \frac{3}{\sqrt{40}} - 128.6}{\frac{3}{\sqrt{40}}}) = P(z < 1.31) = 0.9049$.
Therefore the Type II error is $\beta = 0.0951$.

Part(d).

a. If we decrease the type I error α then the type II error β will increase.

b. If the true population mean is 129.6 volts the type II error will increase (the hypothesized mean is very close to the true mean and therefore it is more difficult to detect a small difference).

Exercise 3

Answer the following questions:

a. The lifetime of certain batteries are supposed to have a variance of 150 hours^2. Using $\alpha = 0.05$ test the following hypothesis
$H_0 : \sigma^2 = 150$
$H_a : \sigma^2 > 150$
if the lifetimes of 15 of these batteries (which constitutes a random sample from a normal population) have:

$$\sum_{i=1}^{15} x_i = 250, \quad \sum_{i=1}^{15} x_i^2 = 8000.$$

where X denotes the lifetime of a battery.

Answer:
We compute first the sample variance s^2:

$$s^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}\right] = \frac{1}{14}\left(8000 - \frac{250^2}{15}\right) = 273.8.$$

The test statistic is: $\frac{(n-1)s^2}{\sigma_0^2} = \frac{(15-1)273.8}{150} = 25.55$. The critical value is $\chi^2_{0.95;14} = 23.68$, therefore H_0 is rejected.

b. A confidence interval is *unbiased* if the expected value of the interval midpoint is equal to the estimated parameter. For example the midpoint of the interval $\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ is $\bar{x}$, and $E(\bar{x}) = \mu$. Now consider the confidence interval for σ^2. Show that the expected value of the midpoint of this confidence interval is not equal to σ^2.

Answer:
The midpoint of the confidence interval for σ^2 is:

$$\frac{1}{2}\left[\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2};n-1}}\right]$$

and its expected value is:

$$\frac{1}{2}E\left[\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2};n-1}}\right] = \frac{1}{2}\left[\frac{(n-1)E(s^2)}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)E(s^2)}{\chi^2_{\frac{\alpha}{2};n-1}}\right].$$

Since $E(s^2) = \sigma^2$ the expression above is:

$$\frac{1}{2}\sigma^2\left[\frac{(n-1)}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)}{\chi^2_{\frac{\alpha}{2};n-1}}\right] \neq \sigma^2$$

because

$$\left[\frac{(n-1)}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)}{\chi^2_{\frac{\alpha}{2};n-1}}\right] \neq 2.$$