# QUIZ 1 SOLUTIONS

(a). $X \sim \Gamma(\frac{1}{2}, 2)$

$$f(x) = \frac{x^{-\frac{1}{2}} e^{-x/2}}{\Gamma(\frac{1}{2})\sqrt{2}}$$

$$Y = X^{1/4}$$

METHOD OF CDF:

$$F_Y(y) = P(Y \le y) = P(X^{1/4} \le y)$$

$$= P(X \le y^4) = F_X(y^4)$$

$$\therefore f_Y(y) = 4y^3 f_X(y^4) = \frac{4y^3 y^{-2} e^{-y^4/2}}{\Gamma(\frac{1}{2})\sqrt{2}} = \frac{4y e^{-y^4/2}}{\Gamma(\frac{1}{2})\sqrt{2}}.$$

(b). $X_1, \ldots, X_n$ iid $\exp(\lambda)$

$$M_{X_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$M_{\Sigma X_i}(t) = (M_{X_i}(t))^n = \left(1 - \frac{t}{\lambda}\right)^{-n}$$

$$\therefore \Sigma X_i \sim \Gamma\left(n, \frac{1}{\lambda}\right)$$

NOTE:

IF $X \sim \Gamma(\alpha, \beta) \to E\, X^k = \dfrac{\Gamma(\alpha+k)\, \beta^k}{\Gamma(\alpha)}$

HERE, $\sum X_i \sim \Gamma(n, \frac{1}{\lambda})$

$$E\, \frac{1}{\sum X_i} = E\left((\sum X_i)^{-1}\right) = \frac{\Gamma(n-1)\left(\frac{1}{\lambda}\right)^{-1}}{\Gamma(n)}$$

$$= \lambda\, \frac{\Gamma(n-1)}{(n-1)\Gamma(n-1)} = \frac{\lambda}{n-1}.$$

$$e^{\ln x^{\alpha}}$$

(c). $f(x) = \dfrac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}$

$$f(x) = \frac{1}{x}\, \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\, e^{\alpha \ln x - \frac{1}{\beta} x}$$

$h(x) = \frac{1}{x}$, $c(\theta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}$ | $w_1(\theta) = \alpha$, $t_1(x) = \ln x$ ; $w_2(\theta) = \frac{1}{\beta}$, $t_2(x) = x$

$$E\left(\sum \frac{dw_i(\theta)}{d\theta_j}\, t_i(x)\right) = -\frac{d \ln(c(\theta))}{d\theta_j}$$

$$\ln c(\theta) = -\alpha \ln \beta - \ln \Gamma(\alpha)$$

$\theta_j = \beta$

$$+\frac{1}{\beta^2} EX = +\left(\frac{\alpha}{\beta}\right) \to EX = \alpha\beta$$

$$\mathrm{var}\left( \sum \frac{d w_i(\theta)}{d\theta_j} t_i(x) \right) = -\frac{d^2 \ln c(\theta)}{d\theta_j^2} - E \sum \frac{d^2 w_i(\theta)}{d\theta_j^2} t_i(x)$$

$$\mathrm{var}\left( \frac{1}{\beta^2} X \right) = -\frac{\alpha}{\beta^2} + E\left( \frac{2\beta}{\beta^4} X \right)$$

$$\frac{1}{\beta^4} \mathrm{var}(X) = -\frac{\alpha}{\beta^2} + \frac{2\alpha\beta^2}{\beta^4} \rightarrow \mathrm{var}(X) = \alpha\beta^2.$$

(d). $M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2} \cdot e^{t\mu + \frac{1}{2}t^2\sigma^2}$

① $\therefore M_{X+Y}(t_1) = e^{t 2\mu + \frac{1}{2} t_1^2 2\sigma^2}$

$M_{X-Y}(t_2) = M_X(t_2) \cdot M_Y(-t_2) = e^{t_2\mu + \frac{1}{2}t_2^2\sigma^2} \cdot e^{-t_2\mu + \frac{1}{2}t_2^2\sigma^2}$

$= e^{\frac{1}{2}t^2 2\sigma^2}$

② $M_{X+Y, X-Y}(t_1, t_2) = E e^{t_1(X+Y) + t_2(X-Y)}$

$= E e^{X(t_1+t_2) + Y(t_1-t_2)}$

$= M_X(t_1+t_2) \cdot M_Y(t_1-t_2)$

$= e^{(t_1+t_2)\mu + \frac{1}{2}(t_1+t_2)^2\sigma^2} \cdot e^{(t_1-t_2)\mu + \frac{1}{2}(t_1-t_2)^2\sigma^2}$

(3). YES, BECAUSE

$$M_{X+Y,\,X-Y}(t_1,t_2) = M_{X+Y}(t_1)\cdot M_{X-Y}(t_2).$$

(e). $X_1, X_2, X_3 \overset{iid}{\sim} N(0,1) \rightarrow \underset{\sim}{X} \sim N_3(\underset{\sim}{0}, \underset{\sim}{I})$

$$\left.\begin{aligned} Y_1 &= X_1 + \delta X_3 \\ Y_2 &= X_2 + \delta X_3 \end{aligned}\right\} \quad \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & \delta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$= \underset{\sim}{A}\,\underset{\sim}{X}$$

THEREFORE, $\underset{\sim}{A}\underset{\sim}{X} \sim N_2(\underset{\sim}{A}\underset{\sim}{\mu}, \underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A}')$

HERE $\underset{\sim}{\mu} = \underset{\sim}{0}$, $\underset{\sim}{\Sigma} = \underset{\sim}{I}$

$$\underset{\sim}{A}\,\underset{\sim}{\Sigma}\,\underset{\sim}{A}' = \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \delta & \delta \end{pmatrix} = \begin{pmatrix} 1+\delta^2 & \delta^2 \\ \delta^2 & 1+\delta^2 \end{pmatrix}$$

$$\rho = \frac{\delta^2}{\sqrt{1+\delta^2}\sqrt{1+\delta^2}} = \frac{1}{2}$$

OR $\dfrac{\delta^2}{1+\delta^2} = \dfrac{1}{2} \rightarrow \delta^2 = 1 \rightarrow \delta = \pm 1.$

(f). FROM LECTURE NOTES:

$$E\left(\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X}\right) = tr\left(\underset{\sim}{A}\underset{\sim}{\Sigma}\right) + \underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu}$$

(g). $Y_1 \mid Y_2, Y_3$ FOLLOWS NORMAL WITH

CONDITIONAL MEAN $\mu_1 + \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}(\underset{\sim}{a}_2 - \underset{\sim}{\mu}_2)$

$$= 170 + \begin{pmatrix} 64 & 128 \end{pmatrix} \begin{pmatrix} 16 & 0 \\ 0 & 256 \end{pmatrix}^{-1} \begin{pmatrix} 72-68 \\ 24-40 \end{pmatrix} = 178$$

AND CONDITIONAL VARIANCE

$$\sigma_1^2 - \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}_{21} = 400 - \begin{pmatrix} 64 & 128 \end{pmatrix} \begin{pmatrix} 16 & 0 \\ 0 & 256 \end{pmatrix}^{-1} \begin{pmatrix} 64 \\ 128 \end{pmatrix} = 80$$

$$\therefore\ Y_1 \mid Y_2, Y_3 \sim N\left(178, \sqrt{80}\right)$$

$$P(Y_1 > 200) = P\left(Z > \frac{200-178}{\sqrt{80}}\right) = P(Z > 2.46)$$

$$= 1 - P(Z < 2.46) = 1 - 0.9931 = 0.0069.$$

(h). DEFINITION

CONTINUOUS

DISCRETE

EXAMPLES: BINOMIAL, POISSON

GAMMA, STANDARD NORMAL

$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$ IF $X, Y$ ARE INDEPENDENT

$M_{X+a}(t)$

$M_{bX}(t)$

$M_{\frac{X+a}{b}}(t) = e^{\frac{a}{b}t} M_X\left(\frac{t}{b}\right)$

USE PROPERTIES TO FIND MGF OF $X \sim N(\mu, \sigma)$

SUM OF iid R.V. $X_1, \ldots, X_n$ iid $N(\mu, \sigma)$

FIND DIST. OF $X_1 + X_2 + \cdots + X_n = \sum X_i$

AND $\bar{X}$

ANOTHER EXAMPLE: $X_1, \ldots, X_n \overset{iid}{\sim} \exp(\lambda)$

$\hookrightarrow \sum X_i \sim \Gamma\left(n, \frac{1}{\lambda}\right)$

USE MGF TO FIND THE MOMENTS:

$M_X'(t)\big|_{t=0} = EX$, $M_X''(t)\big|_{t=0} = EX^2$

OR USE $\psi(t) = \ln M_X(t) \rightarrow$ $\psi'(t)\big|_{t=0} = EX$

$\psi''(t)\big|_{t=0} = VAR(X)$

(i). JOINT MGF

DEFINITION

MOMENTS USING $M_{\underset{\sim}{X}}(\underset{\sim}{t})$

FIND $EX_i$, $VAR(X_i)$, $COV(X_i, X_j)$

USING $M_{\underset{\sim}{X}}(\underset{\sim}{t})$

AND USING $\psi(\underset{\sim}{t}) = \ln M_{\underset{\sim}{X}}(\underset{\sim}{t})$

MARGINAL JOINT MGF

INDEPENDENCE USING JOINT MGF

$\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ THEN $M_{\underset{\sim}{Y}}(\underset{\sim}{t}) = e^{\underset{\sim}{t}'\underset{\sim}{\mu} + \frac{1}{2}\underset{\sim}{t}'\underset{\sim}{\Sigma}\underset{\sim}{t}}$

PLEASE PROVIDE DETAILS ON HOW TO FIND JOINT MGF OF $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

(j). $\theta_1 \sim \Gamma(\alpha_1, \beta_1)$ $\qquad \theta_2 \sim \Gamma(\alpha_2, \beta_2)$

$$E\,\frac{\theta_1}{\theta_2} = E\,\theta_1 \cdot \theta_2^{-1} = E\,\theta_1 \cdot E\,\theta_2^{-1}$$

$$= \alpha_1 \beta_1 \, \frac{\Gamma(\alpha_2 - 1)\, \beta_2^{-1}}{\Gamma(\alpha_2)} = \frac{\alpha_1 \beta_1}{\beta_2 (\alpha_2 - 1)}.$$

WE USE $\quad E\,X^k = \dfrac{\Gamma(\alpha + k)\, \beta^k}{\Gamma(\alpha)}$

FOR THE VARIANCE:

$$\mathrm{VAR}\left(\frac{\theta_1}{\theta_2}\right) = E\left(\frac{\theta_1}{\theta_2}\right)^2 - \left(E\,\frac{\theta_1}{\theta_2}\right)^2$$

$$= E\,\theta_1^2 \cdot E\,\theta_2^{-2} - \left(\frac{\alpha_1 \beta_1}{\beta_2(\alpha_2 - 1)}\right)^2$$

$$= \left(\mathrm{VAR}(\theta_1) + (E\,\theta_1)^2\right) \cdot \frac{\Gamma(\alpha_2 - 2)\, \beta_2^{-2}}{\Gamma(\alpha_2)} - \left(\frac{\alpha_1 \beta_1}{\beta_2(\alpha_2 - 1)}\right)^2$$

$$= \left(\alpha_1 \beta_1^2 + (\alpha_1 \beta_1)^2\right) \frac{1}{(\alpha_2 - 1)(\alpha_2 - 2)\, \beta_2^2} - \left(\frac{\alpha_1 \beta_1}{\beta_2(\alpha_2 - 1)}\right)^2$$