Quiz 2 solutions

EXERCISE 1 :

$$f(u,v) = f(u)\cdot f(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \frac{v^{\frac{n}{2}-1} e^{-v/2}}{\Gamma(\frac{n}{2})\, 2^{n/2}}$$

$$J = \begin{vmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{\frac{v}{n}}} & -\frac{1}{2}u\left(\frac{v}{n}\right)^{-3/2} \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{\frac{w}{n}}}$$

JOINT PDF OF t AND w :

$$f(t,w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2\frac{w}{n}} \frac{w^{\frac{n}{2}-1} e^{-w/2}}{\Gamma(\frac{n}{2})\, 2^{n/2}} \sqrt{\frac{w}{n}}$$

$$f(t,w) = \frac{1}{\sqrt{2\pi n}} \frac{1}{\Gamma(\frac{n}{2})} \frac{1}{2^{n/2}} w^{\frac{n+1}{2}-1} e^{-w\left(\frac{1}{2}\frac{t^2}{n}+\frac{1}{2}\right)}$$

THEREFORE

$$f(t) = \int_0^\infty f(t,w)\,dw = \frac{\Gamma(\frac{n+1}{2})\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \int_0^\infty \frac{w^{\frac{n+1}{2}-1} e^{-\frac{w/2}{(1+\frac{t^2}{n})}}}{\Gamma(\frac{n+1}{2})\left(2/(1+\frac{t^2}{n})\right)^{\frac{n+1}{2}}}\,dw$$

$$= \frac{\Gamma(\frac{n+1}{2})\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})}.$$

EXERCISE 2

$$M_{Q_i}(t) = (1-2t)^{-\frac{p_i}{2}} e^{\theta_i \frac{t}{1-2t}}$$

MOMENT GENERATING FUNCTION OF NON-CENTRAL χ^2 WITH DEGREES OF FREEDOM p_i AND NON CENTRALITY PARAMETER θ_i.

$$M_Y(t) = M_{Q_1}(t) \cdots M_{Q_k}(t)$$

$$= (1-2t)^{-\frac{\sum p_i}{2}} e^{\sum\theta_i \frac{t}{1-2t}}$$

LET $\psi(t) = \ln M_Y(t) = -\frac{\sum p_i}{2} \ln(1-2t) + \frac{t}{1-2t}\sum\theta_i$

$\psi'(t)\big|_{t=0}$ GIVES THE MEAN $\to \sum p_i + \sum\theta_i$

$\psi''(t)\big|_{t=0}$ GIVES THE VARIANCE $\to 2(\sum p_i + 2\sum\theta_i)$

OR $EY = E(Q_1 + \cdots + Q_k)$
$= EQ_1 + \cdots + EQ_k$

AND $VAR(Y) = VAR(Q_1 + \cdots + Q_k)$
$= VAR(Q_1) + \cdots + VAR(Q_k)$

BECAUSE $Q_1, \ldots, Q_k$ ARE INDEPENDENT.

THEN FIND EQ_i AND $VAR(Q_i)$ USING $M_{Q_i}(t)$.

Exercise 3

(a). $M_{\underset{\sim}{X}}(\underline{t}) = \left(p_1 e^{t_1} + p_2 e^{t_2} + \cdots + p_r e^{t_r}\right)^n$

$\therefore M_{X_1}(t_1) = M_{\underset{\sim}{X}}(t_1, 0, 0, \ldots, 0) = \left(p_1 e^{t_1} + p_2 + \cdots + p_r\right)^n$

$= \left(p_1 e^{t_1} + 1 - p_1\right)^n \rightarrow X_1 \sim b(n, p_1)$

(b). $E\left[\underset{\sim}{X}\right] = E\begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = \begin{pmatrix} np_1 \\ \vdots \\ np_r \end{pmatrix}$

VARIANCE COVARIANCE MATRIX OF $\underset{\sim}{X}$:

THE VARIANCES ARE $np_i(1-p_i)$, $i = 1, \ldots, r$

FOR THE COVARIANCES USE THE COROLLARY ON JOINT MGF:

$\psi_i(t) = \dfrac{n p_i e^{t_i}}{p_1 e^{t_1} + \cdots + p_r e^{t_r}}$ AND $\psi_{ij}(\underline{t}) = \dfrac{-p_j e^{t_j} n p_i e^{t_i}}{\left(p_1 e^{t_1} + \cdots + p_r e^{t_r}\right)^2}$

$\therefore \psi_{ij}(\underset{\sim}{0}) = \mathrm{Cov}(X_i, X_j) = -n p_i p_j$.

Exercise 4

$X_i \sim \exp\left(\frac{1}{i\theta}\right) \rightarrow M_{X_i}(t) = (1 - i\theta t)^{-1}$

LET $\hat{\theta} = \sum_{i=1}^{n} \frac{X_i}{ni}$

$M_{\hat{\theta}}(t) = M_{\frac{X_1}{n}}(t) \cdot M_{\frac{X_2}{2n}}(t) \cdots M_{\frac{X_n}{n^2}}(t)$

$$= M_{X_1}\left(\frac{t}{n}\right) \cdot M_{X_2}\left(\frac{t}{2n}\right) \cdots M_{X_n}\left(\frac{t}{n^2}\right)$$

$$= \left(1-\theta\frac{t}{n}\right)^{-1}\left(1-\frac{2\theta t}{2n}\right)^{-1}\cdots\left(1-\frac{n\theta t}{n^2}\right)^{-1}$$

$$= \left(1-\frac{\theta}{n}t\right)^{-n}$$

$$\therefore \quad \hat{\theta} \sim \Gamma\left(n, \frac{\theta}{n}\right)$$

(b). $$E\hat{\theta}^{-1} = \frac{\Gamma(n-1)\left(\frac{\theta}{n}\right)^{-1}}{\Gamma(n)} = \frac{n}{n-1} \cdot \frac{1}{\theta}.$$

(c). $$MSE\left(c\hat{\theta}^{-1}\right) = VAR\left(c\hat{\theta}^{-1}\right) + B^2 = *$$

$$B = E\hat{\theta}^{-1} - \frac{1}{\theta} = \frac{1}{\theta(n-1)}.$$

$$* = c^2\left\{E\left(\hat{\theta}^{-2}\right) - \left(E\hat{\theta}^{-1}\right)^2\right\} + \frac{1}{\theta^2(n-1)^2}$$

$$= c^2\left[\frac{\Gamma(n-2)\left(\frac{\theta}{n}\right)^{-2}}{\Gamma(n)} - \left(\frac{n}{n-1}\frac{1}{\theta}\right)^2\right] + \frac{1}{\theta^2(n-1)^2}$$

THEN $\frac{d\,MSE}{dc} = 0$ AND SOLVE FOR C.

Exercise 5

$$f(x) = \frac{\alpha x^{\alpha-1}}{\theta^{\alpha}}, \quad \alpha > 0, \ \theta > 0$$

$$0 < x < \theta$$

θ IS KNOWN.

$$L = \frac{\alpha^n (x_1 x_2 \dots x_n)^{\alpha-1}}{\theta^{n\alpha}} = \frac{\alpha^n}{\theta^{n\alpha}} \left(\prod x_i\right)^{\alpha-1}$$

LET $g(u, \theta) = \dfrac{\alpha^n}{\theta^{n\alpha}} \left(\prod x_i\right)^{\alpha-1}$

AND $h(x) = 1$

THEREFORE, USING THE FACTORIZATION THEOREM WE CONCLUDE THAT $U = \prod X_i$ IS A SUFFICIENT STATISTIC FOR α.