EXERCISE 1

\[ \mathbb{E}_\mu \hat{\lambda} = \mathbb{E} \left[ \frac{\sum Y}{\sum \mathbb{E} Y} \right] = \frac{\sum Y}{\sum \mathbb{E} Y} = \mu. \]

\[ \text{VAR} \left( \hat{\lambda} \right) = \frac{1}{(\sum \mathbb{E} Y)^2} \text{VAR} \left( \sum \mathbb{E} Y \right) \]

\[ = \frac{1}{\left( \sum \mathbb{E} Y \right)^2} \left( \sum \mathbb{E} Y \right) = \frac{1}{\sum \mathbb{E} Y} \]

WHERE, \( \frac{\mathbb{E} Y}{\sum \mathbb{E} Y} = \frac{1}{\sigma^2(1-p)} \left[ \mathbb{E} - \frac{p}{1+(n-1)p} \right] \)

\[ = \frac{1}{\sigma^2(1-p)} \left( \mathbb{E} - \frac{p}{1+(n-1)p} \right) \]

\[ = \sigma^2(1-p) \left( \frac{n - \frac{p n^2}{1+(n-1)p}}{1+(n-1)p} \right) = \frac{1}{\sigma^2(1-p)} \left( \frac{n+n(n-1)p-n^2p}{1+(n-1)p} \right) \]

\[ = \frac{1}{\sigma^2(1+(n-1)p)}, \text{ THEREFORE } \text{VAR} \left( \hat{\lambda} \right) = \frac{1}{\sigma^2(1+(n-1)p)} > 0 \]

\[ \therefore \rho > \frac{1}{n-1} \]
\textbf{Exercise 2} \quad \gamma \sim N(\theta, \sigma^2)

\[ f(\gamma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\gamma - \theta}{\sigma} \right)^2} \]

\[ L = \left( \frac{1}{2\sigma^2} \right)^n \prod i=1 \gamma_i \]

\[ \ln L = -\frac{n}{2} \ln \sigma^2 + \ln \prod i=1 \gamma_i - \frac{1}{2\sigma^2} \sum i \left( \frac{\gamma_i - \theta}{\sigma} \right)^2 \]

\[ \frac{\partial \ln L}{\partial \theta} = \frac{n}{2\sigma^2} \sum \left( \frac{\gamma_i - \theta}{\sigma} \right) = 0 \]

\[ \hat{\theta} = \frac{1}{n} \sum \gamma_i \]

\[ \text{And} \quad \text{var}(\hat{\theta}) = \frac{1}{n} \sum \frac{\gamma_i^2}{\sigma^2} = \frac{\sigma^2}{n} \]

\textbf{Cramér-Rao Lower Bound}:

\[ \frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{\sigma^4} \]

\[ \frac{1}{-\frac{n}{\sigma^4}} = \frac{\sigma^2}{n} \]

\[ E \left( -\frac{n}{\sigma^4} \right) \]

\[ \hat{\theta} \text{ is \textit{efficient estimator of} } \theta \]
Exercise 4.2.3:

\[ L = (2\pi\sigma^2)^{\frac{n+m}{2}} e^{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2} \]

\[ \ln L = -\frac{n+m}{2} \ln(2\pi) - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2 \]

\[ \frac{d}{d\sigma^2} \ln L = \frac{1}{2\sigma^2} \sum (x_i - \mu)^2 = 0 \]

\[ \sigma^2 = \frac{\sum (x_i - \mu)^2}{n+m} \]
Exercise 4:

\[ y_i = \beta_1 x_i + \epsilon_i \]

\[ y_i \sim N(\beta_1 x_i, \sigma) \]

\[ f(y_i) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} (y_i - \beta_1 x_i)^2} \]

\[ L = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2} \sum (y_i - \beta_1 x_i)^2} \]

\[ \ln L = -\frac{n}{2} \ln(2\pi) - \frac{1}{2\sigma^2} \sum (y_i - \beta_1 x_i)^2 \]

\[ \frac{\partial \ln L}{\partial \beta_1} = -\frac{n}{2\sigma^2} \sum (y_i - \beta_1 x_i) = 0 \]

\[ \hat{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2} = \frac{\sum x_i y_i}{\sum x_i^2} \]

\[ \text{Var} \hat{\beta}_1 = \frac{\sum x_i^2 \text{Var} y_i}{(\sum x_i^2)^2} = \frac{\sum x_i^2 \sigma^2}{(\sum x_i^2)^2} \]

\[ \frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum (y_i - \beta_1 x_i)^2 = 0 \]

\[ \hat{\sigma}^2 = \frac{\sum (y_i - \beta_1 x_i)^2}{n} \]