

Functions of random variables

Functions of one random variable

a. Method of cdf:

Let $X \sim \Gamma(\alpha, \beta)$. Find the distribution of $Y = cX$, $c > 0$. With the method of cdf we begin with the cdf of Y as follows.

$$F_Y(y) = P(Y \leq y)$$

$$F_Y(y) = P(cX \leq y)$$

$$F_Y(y) = P\left(X \leq \frac{y}{c}\right)$$

$$F_Y(y) = F_X\left(\frac{y}{c}\right) \text{ Now differentiate on both sides w.r.t. } y$$

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right)$$

$$f_Y(y) = \frac{1}{c} \frac{\left(\frac{y}{c}\right)^{\alpha-1} e^{-\frac{y}{\beta c}}}{\Gamma(\alpha) \beta^\alpha}$$

$$f_Y(y) = \frac{y^{\alpha-1} e^{-\frac{y}{\beta c}}}{\Gamma(\alpha) (c\beta)^\alpha}$$

Therefore, $Y \sim \Gamma(\alpha, c\beta)$.

Example:

Let $Z \sim N(0, 1)$. Find the pdf of $X = Z^2$.

b. Method of transformations

It is originated from the method of cdf. In general, to find the pdf of a function of a random variable we use the following theorem.

Let X be a continuous random variable with pdf $f(x)$. Let $Y = g(X)$, either increasing or decreasing. Then the pdf of Y is given by

$$f_Y(y) = f_X[w(y)] \left| \frac{d}{dy} w(y) \right|,$$

where $w(y)$ is the inverse function of g (the value of x such that $g(x) = y$). We can also use the following notation, by defining $g^{-1}(y)$ as the value of x such that $g(x) = y$.

$$f_Y(y) = f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right|,$$

Example:

Let $X \sim \Gamma(\alpha, \beta)$. Find the distribution of $Y = cX$, $c > 0$.

$$Y = cX, \text{ here } g(X) = cX, \text{ and therefore } w(y) = g^{-1}(y) = \frac{y}{c}.$$

$$f_Y(y) = f_X[w(y)] \left| \frac{d}{dy} w(y) \right|$$

$$f_Y(y) = f_X\left(\frac{y}{c}\right) \frac{1}{c}$$

$$f_Y(y) = \frac{y^{\alpha-1} e^{-\frac{y}{c\beta}}}{\Gamma(\alpha)(c\beta)^\alpha}$$

A note on the increasing and decreasing requirement of $Y = g(X)$:

c. Method of MGF

Using the uniqueness theorem. Let $X \sim \Gamma(\alpha, \beta)$. Find the distribution of $Y = cX$, $c > 0$. Then

$$M_Y(t) = M_X(ct) = (1 - \beta t)^{-\alpha}. \text{ Therefore, } Y \sim \Gamma(\alpha, c\beta).$$

Joint probability distribution of functions of random variables

We can extend the idea of the distribution of a function of a random variable to bivariate and multivariate random vectors as follows.

Let X_1, X_2 be jointly continuous random variables with pdf $f_{X_1, X_2}(x_1, x_2)$. Suppose $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$. We want to find the joint pdf of Y_1, Y_2 . We follow this procedure:

1. Solve the equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ for x_1 and x_2 in terms of y_1 and y_2 to get $x_1 = h_1(y_1, y_2)$ and $x_2 = h_2(y_1, y_2)$.
2. Compute the Jacobian: $\mathbf{J} = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$. (\mathbf{J} is the determinant of the matrix of partial derivatives.)

To find the joint pdf of Y_1, Y_2 use the following result: $f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2)|\mathbf{J}|^{-1}$, where $|\mathbf{J}|$ is the absolute value of the Jacobian. Here, x_1, x_2 are the expressions obtained from step (1) above, $x_1 = h_1(y_1, y_2)$ and $x_2 = h_2(y_1, y_2)$.

Example 1

Let X_1 and X_2 be independent exponential random variables with parameters λ_1 and λ_2 respectively. Find the joint probability density function of U and V , where $U = X_1 + X_2$ and $V = X_1 - X_2$.

Solution:

Since X_1 and X_2 are independent the joint pdf of X_1 and X_2 is

$$f_{X_1, X_2}(x_1, x_2) =$$

Solve for X_1 and X_2 in terms of U and V :

We compute now the Jacobian:

Finally, we find the joint pdf of U and V :

Example 2

Suppose X and Y are independent random variables with $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$. Compute the joint pdf of $U = X + Y$ and $V = \frac{X}{X+Y}$ and find the distribution of U and the distribution of V . Also show that U, V are independent.

Solution:

A random variable X is said to have a gamma distribution with parameters α, β if its probability density function is given by

$$f(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^\alpha}, \quad \alpha, \beta > 0, x \geq 0.$$

Here $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$, therefore,

$$f_X(x) = \frac{x^{\alpha_1-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha_1) \beta^{\alpha_1}}, \quad \text{and} \quad f_Y(y) = \frac{y^{\alpha_2-1} e^{-\frac{y}{\beta}}}{\Gamma(\alpha_2) \beta^{\alpha_2}}$$

Because X, Y are independent, the joint pdf of X and Y is the product of the two marginal pdfs:

$$f_{XY}(x, y) = f_X(x) f_Y(y) = \frac{x^{\alpha_1-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha_1) \beta^{\alpha_1}} \frac{y^{\alpha_2-1} e^{-\frac{y}{\beta}}}{\Gamma(\alpha_2) \beta^{\alpha_2}} = \frac{x^{\alpha_1-1} y^{\alpha_2-1} e^{-\frac{x+y}{\beta}}}{\Gamma(\alpha_1) \Gamma(\alpha_2) \beta^{\alpha_1+\alpha_2}}.$$

Now follow the two steps above:

1. Solve the equations $u = x + y$ and $v = \frac{x}{x+y}$ in terms of x and y . We get: $x = uv$ and $y = u(1 - v)$.
2. Compute the Jacobian: $\mathbf{J} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \end{vmatrix} = -\frac{1}{x+y} = -\frac{1}{u}$.

Finally to find the joint pdf of U, V use $x = uv$ and $y = u(1 - v)$ in the joint pdf of X, Y :

$f_{UV}(u, v) = \frac{(uv)^{\alpha_1-1} [u(1-v)]^{\alpha_2-1} e^{-\frac{u}{\beta}}}{\Gamma(\alpha_1) \Gamma(\alpha_2) \beta^{\alpha_1+\alpha_2}}$, multiply by $\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$ and rearrange to get :

$$f_{UV}(u, v) = \frac{u^{\alpha_1+\alpha_2-1} e^{-\frac{u}{\beta}}}{\Gamma(\alpha_1 + \alpha_2) \beta^{\alpha_1+\alpha_2}} \times \frac{v^{\alpha_1-1} (1-v)^{\alpha_2-1} \Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)}.$$

Therefore,

$$f_{UV}(u, v) = \frac{u^{\alpha_1+\alpha_2-1} e^{-\frac{u}{\beta}}}{\Gamma(\alpha_1 + \alpha_2) \beta^{\alpha_1+\alpha_2}} \times \frac{v^{\alpha_1-1} (1-v)^{\alpha_2-1}}{B(\alpha_1, \alpha_2)},$$

where, $B(\alpha_1, \alpha_2) = \int_0^1 v^{\alpha_1-1} (1-v)^{\alpha_2-1} dv = \frac{\Gamma(\alpha_1) \Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$ is the Beta function.

We observe that

- a. U, V are independent.
- b. $U \sim \Gamma(\alpha_1 + \alpha_2, \beta)$.
- c. $V \sim \text{Beta}(\alpha_1, \alpha_2)$.

Example 3

Suppose X_1, X_2, X_3 be independent random variables that follow $\Gamma(\alpha_i, 1), i = 1, 2, 3$ distribution. Let

$$\begin{aligned} Y_1 &= \frac{X_1}{X_1 + X_2 + X_3} \\ Y_2 &= \frac{X_2}{X_1 + X_2 + X_3} \\ Y_3 &= \frac{X_3}{X_1 + X_2 + X_3} \end{aligned}$$

denote 3 new random variables. Show that the joint pdf of Y_1, Y_2, Y_3 is given by

$$f(y_1, y_2, y_3) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} y_1^{\alpha_1-1} y_2^{\alpha_2-1} (1 - y_1 - y_2)^{\alpha_3-1}.$$

(Random variables that have a joint pdf of this form follow the Dirichlet distribution.)