Problem 1  (25 points)
Consider the simple regression model
\[ y_i = \beta_0 + \beta_1 x_i + \epsilon_i \]
with \( E(\epsilon_i) = 0,\) \( var(\epsilon_i) = \sigma^2,\) and \( cov(\epsilon_i, \epsilon_j) = 0.\) Answer the following questions:

a. Find \( cov(\epsilon_i, \hat{\beta}_1).\)

b. If \( \beta_0 = 4\) what is the least squares estimate of \( \beta_1? \)

c. What is the variance of the estimate of part (b)?

d. Is the estimate of part (b) unbiased?
Problem 2 (25 points)
Answer the following questions:

a. Consider the simple regression model
\[ y_i = \beta_0 + \beta_1 x_i + \epsilon_i \]
with \( E(\epsilon_i) = 0, \text{var}(\epsilon_i) = \sigma^2, \) and \( \text{cov}(\epsilon_i, \epsilon_j) = 0. \) Show that \( \hat{\beta}_0 \) is BLUE (it has the smallest variance among all the linear unbiased estimators of \( \beta_0 \)).

b. Consider the model of part (a). Find \( \text{cov}(e_i, \hat{Y}_i) \).

c. Consider the simple regression model through the origin
\[ y_i = \beta_1 x_i + \epsilon_i \]
with \( E(\epsilon_i) = 0, \text{var}(\epsilon_i) = \sigma^2, \) and \( \text{cov}(\epsilon_i, \epsilon_j) = 0. \) Show that \( \sum_{i=1}^{n} x_i e_i = 0 \) where \( e_i = Y_i - \hat{Y}_i \).

d. Consider the simple regression model
\[ y_i = \beta_0 + \beta_1 x_i + \epsilon_i \]
with \( E(\epsilon_i) = 0, \text{var}(\epsilon_i) = \sigma^2 x_i, \) and \( \text{cov}(\epsilon_i, \epsilon_j) = 0. \) Also, \( x \) is nonrandom. Is the assumption of constant variance satisfied in the following model? Please explain.
\[ \frac{Y_i}{\sqrt{x_i}} = \frac{\beta_0}{\sqrt{x_i}} + \frac{\beta_1}{\sqrt{x_i}} x_i + \frac{\epsilon_i}{\sqrt{x_i}}. \]
Problem 3 (25 points):
Answer the following questions:

a. Consider the model \( y_i = \beta_0 + \beta_1 x_i + \epsilon_i \). Assume that \( E(\epsilon_i) = 0 \), \( \text{var}(\epsilon_i) = \sigma^2 \), and \( \text{cov}(\epsilon_i, \epsilon_j) = 0 \). Suppose we rescale the \( x \) values as \( x^* = x - \alpha \), and we want to fit the model \( y_i = \beta_0^* + \beta_1^* x_i^* + \epsilon_i \). Find the least squares estimates of \( \beta_0^* \) and \( \beta_1^* \).

b. Refer to the model \( y_i = \beta_0^* + \beta_1^* x_i^* + \epsilon_i \) of part (a). Find the \( SSE \) of this model and compare it to the \( SSE \) of the model \( y_i = \beta_0 + \beta_1 x_i + \epsilon_i \). What is your conclusion?

c. Consider the simple regression model \( y_i = \beta_0 + \beta_1 x_i + \epsilon_i \), with \( E(\epsilon_i) = 0 \), \( \text{var}(\epsilon_i) = \sigma^2 \), and \( \text{cov}(\epsilon_i, \epsilon_j) = 0 \). Show that \( E_{YY} = (n - 2)\sigma^2 + \beta_1^2 S_{XX} \), where \( S_{YY} = \sum_{i=1}^{n}(y_i - \bar{y})^2 \) and \( S_{XX} = \sum_{i=1}^{n}(x_i - \bar{x})^2 \).

d. Refer to the model of part (c). Find \( \text{cov}(\epsilon_i, \epsilon_i) \).
Problem 4  (25 points)
Suppose that a simple linear regression of miles per gallon (Y) on car weight (x) has been performed on 32 observations. The least squares estimates are \( \hat{\beta}_0 = 68.17 \) and \( \hat{\beta}_1 = -1.112 \), with \( s_e = 4.281 \). Other useful information: \( \bar{x} = 30.91 \) and \( \sum_{i=1}^{32} (x_i - \bar{x})^2 = 2054.8 \). Answer the following questions:

a. Construct a 95% confidence interval for \( \beta_1 \).

b. Construct a 95% confidence interval for \( \sigma^2 \).

c. What is the value of \( R^2 \)?

d. Construct a confidence interval for \( 3\beta_0 - 2\beta_1 - 50 \).