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Statistics 100C **Instructor: Nicolas Christou**

Gauss-Markov theorem in multiple regression

In simple regression we showed that among all the linear unbiased estimators of β_0 and β_1 the least squares estimates are BLUE (Best Linear Unbiased Estimators) in the sense that they have the least variance. We will prove now the same theorem when dealing with multiple regression. We will show that if $\mathbf{b}$ is another linear unbiased estimator of $\boldsymbol{\beta}$ its variance covariance matrix will exceed the variance covariance matrix of $\hat{\boldsymbol{\beta}}$ by a positive semidefinite matrix, i.e. $var(\mathbf{b}) \geq var(\hat{\boldsymbol{\beta}})$.

Proof
The OLS estimates are given by $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\mathbf{Y}$. Let $\mathbf{b} = \mathbf{M}^*\mathbf{Y}$ be another linear unbiased estimator of $\boldsymbol{\beta}$. Let's define $\mathbf{M}^* = \mathbf{M} + (\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'$. Since $\mathbf{b}$ is unbiased it follows that $E\mathbf{b} = \boldsymbol{\beta}$. Show that $\mathbf{MX} = \mathbf{0}$.

Now let's examine the variance of $\mathbf{b}$. Show that $var(\mathbf{b}) = \text{something} + var(\hat{\boldsymbol{\beta}})$. Don't forget that $\mathbf{MX} = \mathbf{0}$.

$$
\begin{aligned}
var(\mathbf{b}) &= var(\mathbf{M}^*\mathbf{Y}) = var\left[\mathbf{M} + (\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\right]\mathbf{Y} \\
&= \\
&=
\end{aligned}
$$

We see that $var(\mathbf{b})$ is reduced to $var(\mathbf{b}) = \sigma^2\mathbf{MM}' + \sigma^2(\mathbf{X}'\mathbf{X})^{-\mathbf{1}} = \sigma^2\mathbf{MM}' + var(\hat{\boldsymbol{\beta}})$. It remains to show that $\mathbf{MM}'$ is positive semidefinite. Why is this true?

The Gauss-Markov theorem for a linear combination of the vector $\hat{\boldsymbol{\beta}}$.
Let $\mathbf{a}'\hat{\boldsymbol{\beta}}$ be a linear combination of $\hat{\boldsymbol{\beta}}$. Its variance is equal to $var(\mathbf{a}'\hat{\boldsymbol{\beta}}) = \sigma^2\mathbf{a}'(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{a}$. Why?

Now let $\mathbf{a}'\mathbf{b}$ be another unbiased estimator of $\mathbf{a}'\boldsymbol{\beta}$. Show that $var(\mathbf{a}'\mathbf{b}) \geq var(\mathbf{a}'\hat{\boldsymbol{\beta}})$.

Suppose $\mathbf{a} = (0, 0, \ldots, 1, 0, \ldots, 0)'$? What do we conclude about the variance of b_i compared to the variance of $\hat{\beta}_i$?