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Department of Statistics

Statistics 100C **Instructor: Nicolas Christou**

Homework 1

Exercise 1
Suppose that we want to test the following two hypotheses:
$H_0 : \mu_1 - \mu_2 = 0$
$H_a : \mu_1 - \mu_2 > 0$
To perform this test a sample of n observations from each population is to be selected. Assume that the populations are normally distributed with known variances σ_1^2 and σ_2^2 respectively.

a. If we require $1-\beta$ power of the test and we are willing to accept Type I error α, find an expression for the sample size needed to detect a shift in the difference between the two population means from $\mu_1 - \mu_2 = 0$ to $\mu_1 - \mu_2 = \delta$ ($\delta > 0$).

b. Use the result from part (a) to find n when $\sigma_1 = 12, \sigma_2 = 15, \delta = 3, \alpha = 0.01$, and $1 - \beta = 0.99$.

Exercise 2
Let $X_1, X_2, \cdots, X_9$ and $Y_1, Y_2, \cdots, Y_{12}$ represent two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$. It is given that $\sigma_1^2 = 3\sigma_2^2$, but σ_2^2 is unknown. Define a random variable which has a t distribution and use it to find a 95% confidence interval for $\mu_1 - \mu_2$.

Exercise 3
Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(0, \sigma)$.

a. Show that $\sum_{i=1}^n X_i^2 > k'$ is the best critical region for testing
$H_0 : \sigma^2 = 4$
$H_a : \sigma^2 = 16$
using the Neyman-Pearson lemma.

b. If $n = 15$, find the value of k' so that $\alpha = 0.05$.

c. If $n = 15$ and k' is the value found in part (b), compute the value of

$$\beta = P(\sum_{i=1}^n X_i^2 < k' \quad \text{when} \quad \sigma^2 = 16).$$

Exercise 4
Suppose that $X_1, X_2, \cdots, X_n$ form a random sample from a normal distribution for which both the mean μ and the variance σ^2 are unknown. A confidence interval for μ is to be constructed with confidence level 90%. Determine the smallest value of n such that the expected squared length of this interval will be less than $\frac{\sigma^2}{2}$.

Exercise 5
Let $X_1, X_2, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$, with known mean μ. Describe how you would construct a confidence interval for the unknown variance σ^2 with confidence level $1 - \alpha$.

Exercise 6
We noted that if we sample from a normal population

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}$$

is a biased estimator of σ^2 and that

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$$

is an unbiased estimator of σ^2.

a. Find $Var(\hat{\sigma}^2)$.

b. Show that $Var(S^2) > Var(\hat{\sigma}^2)$.

Exercise 7
Suppose $Y_i = \beta x_i + \epsilon_i$. In this equation X is non-random, β is a parameter (unknown), and $\epsilon \sim N(0, \sigma)$.

a. Find the mean of Y.

b. Find the variance of Y.

c. What distribution does Y follow?

d. Write down the likelihood function based on n observations of Y and x.

e. Find the maximum likelihood estimate of β. Denote it with $\hat{\beta}$.

f. Show that the estimate of part (e) is unbiased estimator of β.

g. Find the variance of this estimate.

Exercise 8
Consider the model of exercise 7. Find the covariance between e_i and e_j. Also find the variance of e_i.