Exercise 1
Consider the models
\[ Y = X\beta + \epsilon, \quad Y^* = X^*\beta + \epsilon^*, \]
where \( E(\epsilon) = 0, \) \( \text{cov}(\epsilon) = \sigma^2 I, \) \( Y^* = \Gamma Y, X^* = \Gamma X, \epsilon^* = \Gamma \epsilon, \) and \( \Gamma \) is a known \( n \times n \) orthogonal matrix. Here orthogonal means \( \Gamma' \Gamma = I. \) Show that:

a. \( E(\epsilon^*) = 0, \) and \( \text{cov}(\epsilon^*) = \sigma^2 I. \)

b. Show that \( \hat{\beta} = \hat{\beta}^* \) and \( s^2_e = s^*_{e^2}, \) where \( \hat{\beta} \) and \( \hat{\beta}^* \) are the least squares estimates of \( \beta \) of the two models.

Exercise 2
Show that when we do weighted least squares instead of weighting by \( \frac{1}{c_1^2}, \ldots, \frac{1}{c_n^2}, \) we had weighted by \( \frac{a}{c_1^2}, \ldots, \frac{a}{c_n^2} \) where \( a \) is any positive number, then \( \hat{\beta}_{WLS} \) and its variance-covariance matrix are unaffected.

Exercise 3
For the simple regression model without intercept \( y_i = \beta x_i + \epsilon_i \) assume that \( E(\epsilon_i) = 0, E(\epsilon_i \epsilon_j) = 0, \) and \( \text{var}(\epsilon_i) = \sigma^2 x_i. \) Find the weighted least squares estimate of \( \beta \) and obtain its variance.

Exercise 4
Consider the regression model through the origin \( y_i = \beta x_i + \epsilon_i \) with \( E(\epsilon_i) = 0, E(\epsilon_i \epsilon_j) = 0, \) and \( \text{var}(\epsilon_i) = \sigma^2 x_i^2. \) Derive the weighted least squares estimate of \( \beta \) and obtain its variance.

Exercise 5
Answer the following questions:

a. Assume that the columns of the matrix \( X \) in the model \( Y = X\beta + \epsilon, \epsilon \sim \text{MVN}(0, \sigma^2 I) \) are orthogonal. Show that \( \hat{\beta}_i \) and \( \hat{\beta}_j \) are independent.

b. Suppose an extra term is added to the model. So the model now is:
\[ Y = X\beta + \gamma z + \epsilon. \]
If \( z \) is orthogonal to \( X \) show that the estimate of \( \beta \) in this model is the same as the estimate of the model of part (a).