LIKELIHOOD RATIO TEST :

$$y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

TEST $\left.\begin{array}{l} H_0: \beta_1 = 1 \\ H_a: \beta_1 \neq 1 \end{array}\right\}$

UNDER H_0 : $y_i = \beta_0 + X_i + \varepsilon_i$

$$y_i \sim N(\beta_0 + X_i, \sigma)$$

$$L = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(y_i - \beta_0 - X_i)^2}$$

$$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum[y_i - \beta_0 - X_i]^2$$

$$\frac{\partial \ln L}{\partial \beta_0} = +\frac{2}{2\sigma^2}\sum[y_i - \beta_0 - X_i] = 0$$

$$\sum y_i - n\beta_0 - \sum X_i = 0 \rightarrow \hat{\beta}_0 = \bar{Y} - \bar{X}.$$

$$\frac{\partial \ln L}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(y_i - \beta_0 - X_i)^2 = 0$$

$$\hat{\sigma}_0^2 = \frac{\sum[y_i - \hat{\beta}_0 - X_i]^2}{n}$$

OR $$\hat{\sigma}_0^2 = \frac{\sum[y_i - \bar{y} - (X_i - \bar{X})]^2}{n}$$

UNDER NO RESTRICTIONS WE GET THE OLS ESTIMATES $\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}_1^2$

LRT: $$\frac{L(\hat{\omega})}{L(\hat{\Omega})} < K$$

$$\frac{(2\pi\hat{\sigma}_0^2)^{-n/2}\, e^{-\frac{1}{2\hat{\sigma}_0^2}\overbrace{\sum(y_i-\hat{\beta}_0-x_i)^2}^{n\hat{\sigma}_0^2}}}{(2\pi\hat{\sigma}_1^2)^{-n/2}\, e^{-\frac{1}{2\hat{\sigma}_1^2}\underbrace{\sum(y_i-\hat{\beta}_0-\hat{\beta}_1x_i)^2}_{n\hat{\sigma}_1^2}}} < K$$

$$\frac{(2\pi\hat{\sigma}_0^2)^{-n/2}}{(2\pi\hat{\sigma}_1^2)^{-n/2}}\,\frac{\cancel{e^{-n/2}}}{\cancel{e^{-n/2}}} < K$$

$$\frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2} < K^{2/n}$$

OR $$\frac{\sum e_i^2}{\sum[y_i - \bar{y} - (x_i-\bar{x})]^2} < K^{2/n}$$

$\pm \hat{y}_i$

EXAMINE THE DENOMINATOR:

$$\sum\Big(\underbrace{y_i-\hat{y}_i}_{e_i} + \underset{\substack{\downarrow \\ \bar{y}+\hat{\beta}_1(x_i-\bar{x})}}{\hat{y}_i} - \bar{y} - (x_i-\bar{x})\Big]^2$$

$$= \sum e_i^2 + \hat{\beta}^2\sum(x_i-\bar{x})^2 + \sum(x_i-\bar{x})^2 - 2\hat{\beta}_1\sum(x_i-\bar{x})^2$$

(THE OTHER TWO TERMS ARE ZERO).

$$= \sum e_i^2 + (\hat{\beta}_1-1)^2\sum(x_i-\bar{x})^2$$

BACK TO THE LRT:

$$\frac{\sum e_i^2}{\sum e_i^2 + (\hat{\beta}_1-1)^2\sum(x_i-\bar{x})^2} < k^{2/n}$$

DIVIDE NUMERATOR AND DENOMINATOR BY $\sum e_i^2$:

$$\frac{1}{1+\dfrac{(\hat{\beta}_1-1)^2\sum(x_i-\bar{x})^2}{\sum e_i^2}} < k^{2/n}$$

OR

$$\frac{(\hat{\beta}_1-1)^2\sum(x_i-\bar{x})^2}{(n-2)s_e^2} > k^{-2/n}-1$$

$$\frac{(\hat{\theta}_1 - 1)^2 \sum (x_i - \bar{x})^2}{S_e^2} > \left[k^{-2/n} - 1 \right](n-2)$$

NOW SHOW THAT THIS FOLLOWS $F_{1, n-2}$ UNDER H_0.

UNDER H_0: $\hat{\theta}_1 \sim N\left(1, \frac{\sigma}{\sqrt{\sum (x_i - \bar{x})^2}}\right)$

$$\frac{\hat{\theta}_1 - 1}{\sigma / \sqrt{\sum (x_i - \bar{x})^2}} \sim N(0, 1)$$

$$\therefore \left.\begin{array}{l} \dfrac{(\hat{\theta}_1 - 1)^2 \sum (x_i - \bar{x})^2}{\sigma^2} \sim \chi^2_1, \\ \text{ALSO, } \dfrac{(n-2) S_e^2}{\sigma^2} \sim \chi^2_{n-2} \end{array}\right\}$$

THESE TWO DISTRIBUTIONS WILL GIVE:

$$\frac{(\hat{\theta}_1 - 1) \sum (x_i - \bar{x})^2}{S_e^2} \sim F_{1, n-2}$$

SAME AS WITH THE LRT.