University of California, Los Angeles
Department of Statistics

Statistics 100C **Instructor: Nicolas Christou**

Quiz 1

You have 2 hours to answer the questions and upload the solutions on Gradescope. Thank you!

Answer the following questions:

a. Consider the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, ..., n$. The Gauss-Markov conditions hold. In class we showed that $E[\hat{\beta}_1] = \beta_1$ and $\text{var}[\hat{\beta}_1] = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$ using the results on $k_i's$. Here we want to find the expected value and variance of $\hat{\beta}_1$ directly, without the $k_i's$. In other words you should begin your derivation as follows

$$E[\hat{\beta}_1] = E\left[\frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2}\right] = \ldots$$

$$\text{var}[\hat{\beta}_1] = \text{var}\left[\frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2}\right] = \ldots$$

b. Refer to question (a). Do the same for $E[\hat{\beta}_0]$ and $\text{var}[\hat{\beta}_0]$ without using the $l_i's$ that we discussed in class. Here, you should begin your derivation as follows:

$$E[\hat{\beta}_0] = E\left[\bar{Y} - \hat{\beta}_1 \bar{x}\right] = \ldots$$

$$\text{var}[\hat{\beta}_0] = \text{var}\left[\bar{Y} - \hat{\beta}_1 \bar{x}\right] = \ldots$$

Please note that for the variance of $\hat{\beta}_0$ you will need the following result: $\text{cov}[\bar{Y}, \hat{\beta}_1] = 0$.

c. Consider the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, ..., n$. The Gauss-Markov conditions hold. Show that $\sum_{i=1}^n y_i^2 = \sum_{i=1}^n \hat{y}_i^2 + \sum_{i=1}^n e_i^2$.

d. Consider the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, ..., n$. The Gauss-Markov conditions hold. Show that $\sum_{i=1}^n (\hat{y}_i - \bar{y})^2 = \hat{\beta}_1^2 (n-1) S_x^2$, where S_x^2 is the sample variance of $x_1, \ldots, x_n$.

e. Consider the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, ..., n$. Assume that $E[\epsilon_i] = 0, \text{var}[\epsilon_i] = \sigma^2$, and $\text{cov}(\epsilon_i, \epsilon_j) = \rho \sigma^2$ (therefore, $\epsilon_1, \ldots, \epsilon_n$ are not independent.) Consider the OLS estimators $\hat{\beta}_1$ and $\hat{\beta}_0$. Are they still unbiased?

f. Consider the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, ..., n$. The Gauss-Markov conditions hold. Suppose each y_i is multiplied by the same constant c and each x_i is multiplied by the same constant d. Express $\hat{\beta}_1$ and $\hat{\beta}_0$ of the transformed model in terms of $\hat{\beta}_1$ and $\hat{\beta}_0$ of the original model.

g. Consider the simple regression model $y_i = \delta_0 + \delta_1 (x_i - \bar{x}) + \epsilon_i, i = 1, ..., n$. The Gauss-Markov conditions hold. In addition assume that $\epsilon_i \sim N(0, \sigma)$. Find the maximum likelihood estimates of δ_0, δ_1, and σ^2.

h. Refer to question (g). What is the distribution of $\hat{\delta}_0$ and $\hat{\delta}_1$?

i. Consider the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, ..., n$. The Gauss-Markov conditions hold. In addition assume that $\epsilon_i \sim N(0, \sigma)$. We have seen in lecture that the maximum likelihood estimator of σ^2 is $\hat{\sigma^2} = \frac{\sum_{i=1}^n e_i^2}{n}$. Is it true that $E[\hat{\sigma^2}] = \frac{1}{n} \sum_{i=1}^n \text{var}[e_i]$?

j. Consider the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, ..., n$. The Gauss-Markov conditions hold. In addition assume that $\epsilon_i \sim N(0, \sigma)$. We have seen in lecture that $\sum_{i=1}^n e_i = 0$. What can we conclude about $\sum_{i=1}^n \epsilon_i$? What is the variance of $\bar{\epsilon} = \frac{1}{n} \sum_{i=1}^n \epsilon_i$? Find the distribution of $\bar{\epsilon}$ and the distribution of $\bar{Y}$.

Good luck!