**University of California, Los Angeles**
**Department of Statistics**

**Statistics 100C** **Instructor: Nicolas Christou**

**Quiz 2**

You have 2 hours to answer the questions and upload the solutions on Gradescope. Thank you!

a. Consider the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, \ldots, n$. The Gauss-Markov conditions hold. In addition assume that $\epsilon_i \sim N(0, \sigma), i = 1, \ldots, n$. Use the distribution of $Y_i, i = 1, \ldots, n$ to create a $\chi^2$ random variable with $n$ degrees of freedom.

b. In class we showed using the centered model that $\frac{(n-2)S_e^2}{\sigma^2} \sim \chi^2_{n-2}$. Here we will use a different approach. Refer to question (a). Inside the bracket of the expression $\sum_{i=1}^n \left[\frac{Y_i - \beta_0 - \beta_1 x_i}{\sigma}\right]^2$ add/subtract $\hat{\beta}_0$ and add/subtract $\hat{\beta}_1 x_i$ and expand to show that
$\sum_{i=1}^n \left[\frac{Y_i - \beta_0 - \beta_1 x_i}{\sigma}\right]^2 = \frac{\sum_{i=1}^n e_i^2}{\sigma^2} + \frac{n(\hat{\beta}_0 - \beta_0)^2}{\sigma^2} + \frac{(\hat{\beta}_1 - \beta_1)^2 \sum_{i=1}^n x_i^2}{\sigma^2} + \frac{2(\hat{\beta}_0 - \beta_0)(\hat{\beta}_1 - \beta_1)\sum_{i=1}^n x_i}{\sigma^2}$.

c. Define now, $Q = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$, therefore $Q = \bar{Y}$. Consider the expression $\frac{(\hat{\beta}_1 - \beta_1)^2}{\text{var}(\hat{\beta}_1)} + \frac{[Q - (\beta_0 + \beta_1 \bar{x})]^2}{\text{var}[Q]}$. Expand this expression to show that it is equal to the last three terms of the right hand side of the equation in (b). Therefore show that $\frac{(\hat{\beta}_1 - \beta_1)^2}{\text{var}(\hat{\beta}_1)} + \frac{[Q - (\beta_0 + \beta_1 \bar{x})]^2}{\text{var}[Q]} = \frac{n(\hat{\beta}_0 - \beta_0)^2}{\sigma^2} + \frac{(\hat{\beta}_1 - \beta_1)^2 \sum_{i=1}^n x_i^2}{\sigma^2} + \frac{2(\hat{\beta}_0 - \beta_0)(\hat{\beta}_1 - \beta_1)\sum_{i=1}^n x_i}{\sigma^2}$.

d. Therefore back to the equation in part (b) we have:

$$\begin{array}{ccccccc} \sum_{i=1}^n \left[\frac{Y_i - \beta_0 - \beta_1 x_i}{\sigma}\right]^2 & = & \frac{\sum_{i=1}^n e_i^2}{\sigma^2} & + & \frac{(\hat{\beta}_1 - \beta_1)^2}{\text{var}(\hat{\beta}_1)} & + & \frac{[Q - (\beta_0 + \beta_1 \bar{x})]^2}{\text{var}[Q]} \\ Q & = & Q_1 & + & Q_2 & + & Q_3 \end{array}$$

Are the three terms $Q_1, Q_2, Q_3$ independent? Why?

e. Refer to question (d). What is the distribution of $Q_2 + Q_3$? Conclude that $Q_1 = \frac{(n-2)S_e^2}{\sigma^2} \sim \chi^2_{n-2}$.

f. Refer to question (a). Use the pairing method to find $\text{cov}[\hat{\beta}_0, e_i]$. Are $\hat{\beta}_0$ and $S_e^2$ independent?

g. Refer to question (a). Find $E[\hat{\beta}_1 - \hat{\beta}_0]$ and $\text{var}[\hat{\beta}_1 - \hat{\beta}_0]$.

h. Refer to question (a). $\text{cov}[\hat{\beta}_1 - \hat{\beta}_0, e_i]$.

i. Consider the simple regression model $y_i = \beta_1 x_i + \epsilon_i, i = 1, \ldots, n$. The Gauss-Markov conditions hold. In addition assume that $\epsilon_i \sim N(0, \sigma), i = 1, \ldots, n$. Find $\text{cov}(\hat{Y}_i, \hat{Y}_j)$ and $\text{cov}(e_i, e_j)$.

j. Refer to question (i). Find $\text{cov}(\hat{\beta}_1, e_i)$.

Good luck!