Bonferroni confidence intervals

Suppose we want to find confidence intervals \( I_1, I_2, \ldots, I_m \) for parameters \( \theta_1, \theta_2, \ldots, \theta_m \). In the analysis of variance problem \( \theta_1 = \mu_1 - \mu_2, \theta_2 = \mu_1 - \mu_3, \ldots, \theta_m = \mu_{k-1} - \mu_k \). We want
\[
P(\theta_j \in I_j, j = 1, 2, \ldots, m) \geq 1 - \alpha,
\]
i.e. we want a simultaneous confidence level \( 1 - \alpha \).

Aside note:
De Morgan’s law states that:
\[
P(A_1 \cap A_2 \cap \ldots \cap A_m) = 1 - P(A_1' \cup A_2' \cup \ldots \cup A_m')
\]
Bonferroni inequality:
\[
P(A_1' \cup A_2' \cup \ldots \cup A_m') \leq \sum_{i=1}^{m} P(A_i')
\]
Therefore,
\[
P(A_1 \cap A_2 \cap \ldots \cap A_m) \geq 1 - \sum_{i=1}^{m} P(A_i') \quad (1)
\]
Suppose \( P(\theta_j \in I_j) = 1 - \alpha_j \). Let’s denote with \( A_j \) the event that \( \theta_j \in I_j \). Then from Bonferroni inequality (see expression (1) above) we get:
\[
P(\theta_1 \in I_1, \theta_2 \in I_2, \ldots, \theta_m \in I_m) \geq 1 - \sum_{i=1}^{m} P(\theta_j \notin I_j)
\]
\[
\geq 1 - \sum_{i=1}^{m} \alpha_j.
\]
If all \( \alpha_j, j = 1, 2, \ldots, m \) are chosen equal to \( \alpha \) we observe that the simultaneous confidence level is only \( \geq 1 - m\alpha \) which is smaller than \( 1 - \alpha \) because \( m \geq 1 \). In order to have a simultaneous confidence interval \( 1 - \alpha \) we will need to use confidence level \( 1 - \frac{\alpha}{m} \) for each confidence interval for \( \mu_1 - \mu_2, \mu_1 - \mu_3, \ldots, \mu_{k-1} - \mu_k \).

Bonferroni confidence interval for \( \mu_i - \mu_j \):
\[
\bar{x}_i - \bar{x}_j \pm t_{\frac{\alpha}{m}; N-k} s_p \sqrt{\frac{1}{n_i} + \frac{1}{n_j}},
\]
where \( s_p \) is the estimate of the common but unknown standard deviation:
\[
s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \ldots + (n_k - 1)s_k^2}{N-k}}
\]
Example (see ANOVA handout):
Construct a Bonferroni confidence interval for \( \mu_1 - \mu_4 \) using 95% simultaneous confidence level. The data are \( \bar{x}_1 = 4.062, \bar{x}_4 = 3.920, n_1 = 10, n_2 = 10, s_p = 0.06061 \). We also need \( t_{0.05;29} = t_{0.025;29} = t_{0.00119;63} = 3.1663 \).
\[
\mu_1 - \mu_4 \in 4.062 - 3.920 \pm 3.1663(0.06061)\sqrt{\frac{1}{10} + \frac{1}{10}},
\]
or
\[
\mu_1 - \mu_4 \in 0.142 \pm 0.086,
\]
or
\[
0.056 \leq \mu_1 - \mu_4 \leq 0.228.
\]