Discussion 1 solutions

(a). $\sum y_i^2 = \sum (y_i - \hat{y}_i + \hat{y}_i)^2 = \sum [e_i + \hat{y}_i]^2$

$= \sum e_i^2 + \sum \hat{y}_i^2 + 2 \underbrace{\sum e_i \hat{y}_i}_{0} = \sum e_i^2 + \sum \hat{y}_i^2$

(b). $VAR(\bar{Y} + \hat{\beta}_1) = VAR(\bar{Y}) + VAR(\hat{\beta}_1) + 2\underbrace{Cov(\bar{Y}, \hat{\beta}_1)}_{0}$

$= \frac{\sigma^2}{n} + \frac{\sigma^2}{\sum (x_i - \bar{x})^2} = \sigma^2 \left(\frac{1}{n} + \frac{1}{\sum (x_i - \bar{x})^2} \right)$

(c) $E(\bar{Y} + \hat{\beta}_1)^2 = VAR(\bar{Y} + \hat{\beta}_1) + \{E(\bar{Y} + \hat{\beta}_1)\}^2$

$= \sigma^2 \left(\frac{1}{n} + \frac{1}{\sum (x_i - \bar{x})^2} \right) + \{\beta_0 + \beta_1 \bar{x} + \beta_1\}^2$

(d). $\hat{y}_i = \bar{y} + \hat{\beta}_1 (x_i - \bar{x}) = \frac{\sum y_j}{n} + (x_i - \bar{x}) \sum k_j y_j$

$= \sum \left(\frac{1}{n} + (x_i - \bar{x}) k_j \right) y_j = \sum_{j=1}^{n} h_{ij} y_j$

$\therefore h_{ij} = \frac{1}{n} + (x_i - \bar{x}) k_j$

$\sum h_{ij} = \sum \left(\frac{1}{n} + (x_i - \bar{x}) k_j \right) = 1 + (x_i - \bar{x}) \underbrace{\sum k_j}_{0}$

$= 1.$

(e). $\hat{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2}$ $\quad \hat{y}_i = \hat{\beta}_1 x_i$, $\hat{y}_j = \hat{\beta}_1 x_j$

$$\text{Cov}(e_i, e_j) = \text{Cov}(y_i - \hat{\beta}_1 x_i,\ y_j - \hat{\beta}_1 x_j)$$

$$= \text{Cov}(y_i, y_j) - x_j \text{Cov}(y_i, \hat{\beta}_1)$$

$$- x_i \text{Cov}(\hat{\beta}_1, y_j) + x_i x_j \text{VAR}(\hat{\beta}_1)$$

$$= 0 - x_j x_i \frac{\sigma^2}{\sum x_i^2}$$

$$- x_i x_j \frac{\sigma^2}{\sum x_i^2} + x_i x_j \frac{\sigma^2}{\sum x_i^2} = - \frac{x_i x_j \sigma^2}{\sum x_i^2}$$

NOTE: $\text{VAR}(\hat{\beta}_1) = \frac{\sigma^2}{\sum x_i^2}$

(f). $\sum y_i^2 = \sum (y_i - \hat{y}_i + \hat{y}_i)^2 = \sum (e_i + \hat{y}_i)^2$

$$= \sum e_i^2 + \sum \hat{y}_i^2 + 2 \sum e_i \hat{y}_i$$

<u>NOTE</u>: $\sum e_i \hat{y}_i = \sum (y_i - \hat{\beta}_1 x_i) \hat{\beta}_1 x_i$

$$= \hat{\beta}_1 \sum x_i y_i - \hat{\beta}_1^2 \sum x_i^2 = 0.$$

ALSO, $\hat{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2}$ $\qquad$ <u>YES</u>.

(g). $\hat{y}_i = \hat{\beta}_1 x_i = \frac{\sum x_i x_j y_j}{\sum x_j^2}$

$= \sum h_{ij} y_j$ WHERE, $h_{ij} = \frac{x_i x_j}{\sum x_j^2}$

NOW $\sum h_{ij} \neq 1$

(h) $y_i = \beta_1 x_i + \varepsilon_i$

$c_1 y_i = c_2 \beta_1 x_i + \varepsilon_i$

$y_i^* = \beta_1 x_i^* + \varepsilon_i$

$\hat{\beta}_{1\,NEW} = \frac{\sum x_i^* y_i^*}{\sum x_i^{*2}} = \frac{c_1 c_2 \sum x_i y_i}{c_2^2 \sum x_i^2} = \frac{c_1}{c_2} \hat{\beta}_{1\,OLD}$