The fitted values and their variance-covariance matrix

The variance-covariance matrix of the fitted values can be expressed as follows:

$$\text{cov}(\hat{Y}) = \text{cov}(X\hat{\beta}) = X\text{cov}(\hat{\beta})X' = \sigma^2 X(X'X)^{-1}X' = \sigma^2 H.$$

Or if we expand this we get:

$$\text{cov}(\hat{Y}) = \begin{pmatrix}
\text{var}(\hat{y}_1) & \text{cov}(\hat{y}_1, \hat{y}_2) & \text{cov}(\hat{y}_1, \hat{y}_3) & \cdots & \cdots & \text{cov}(\hat{y}_1, \hat{y}_n) \\
\text{cov}(\hat{y}_2, \hat{y}_1) & \text{var}(\hat{y}_2) & \text{cov}(\hat{y}_2, \hat{y}_3) & \cdots & \cdots & \text{cov}(\hat{y}_2, \hat{y}_n) \\
\cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\
\text{cov}(\hat{y}_n, \hat{y}_1) & \text{cov}(\hat{y}_n, \hat{y}_2) & \text{cov}(\hat{y}_n, \hat{y}_3) & \cdots & \cdots & \text{var}(\hat{y}_n)
\end{pmatrix} \Rightarrow$$

$$\text{cov}(\hat{Y}) = \sigma^2 \begin{pmatrix}
h_{11} & h_{12} & h_{13} & \cdots & \cdots & h_{1n} \\
h_{21} & h_{22} & h_{23} & \cdots & \cdots & h_{2n} \\
\cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\
h_{n1} & h_{n2} & h_{n3} & \cdots & \cdots & h_{nn}
\end{pmatrix}$$

Where, $h_{ij}$ is the $ij$th element of the hat matrix $H$. Therefore the variance of the $i$th fitted value is $\text{var}(\hat{y}_i) = \sigma^2 h_{ii}$. Since the variance is always $\geq 0$ it means that $h_{ii} \geq 0$. Also from a previous discussion we found that $h_{ii} \leq 1$. Therefore we have $\frac{1}{n} \leq h_{ii} \leq 1$.

The sum of the diagonal elements of the hat matrix is equal to $k + 1$ (in simple regression $k = 1 \Rightarrow \sum_{i=1}^{n} h_{ii} = 2$).

Proof: The trace of a square matrix is equal to the sum of its diagonal elements. Also a property of the trace is the following: Let $A, B, C$ be matrices. Then $\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$. We can apply this result to the hat matrix:

$$\text{tr} \left( X(X'X)^{-1}X' \right) = \text{tr} \left( XX'(XX')^{-1} \right) = \text{tr} \left( I_{(k+1)\times(k+1)} \right) = k + 1.$$

For simple regression it can be shown that

$$h_{ii} = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

You can easily verify that $\sum_{i=1}^{n} h_{ii} = 2$. 
