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Statistics 100C **Instructor: Nicolas Christou**

Joint probability distribution of functions of random variables

Let X_1, X_2 be jointly continuous random variables with pdf $f_{X1X2}(x_1, x_2)$. Suppose $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$. We want to find the joint pdf of Y_1, Y_2. We follow this procedure:

1. Solve the equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ for x_1 and x_2 in terms of y_1 and y_2.
2. Compute the Jacobian: $\mathbf{J} = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$. ($\mathbf{J}$ is the determinant of the matrix of partial derivatives.)

To find the joint pdf of Y_1, Y_2 use the following result: $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|\mathbf{J}|^{-1}$, where $|\mathbf{J}|$ is the absolute value of the Jacobian. Here, x_1, x_2 are the expressions obtained from step (1) above.

Example:
Suppose X and Y are independent random variables with $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$. Compute the joint pdf of $U = X + Y$ and $V = \frac{X}{X+Y}$ and find the distribution of U and the distribution of V. Also show that U, V are independent.

Solution:
A random variable X is said to have a gamma distribution with parameters α, β if its probability density function is given by

$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}, \quad \alpha, \beta > 0, x \geq 0.$$

Here $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$, therefore,

$$f_X(x) = \frac{x^{\alpha_1-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha_1)\beta^{\alpha_1}}, \text{ and } f_Y(y) = \frac{y^{\alpha_2-1}e^{-\frac{y}{\beta}}}{\Gamma(\alpha_2)\beta^{\alpha_2}}$$

Because X, Y are independent, the joint pdf of X and Y is the product of the two marginal pdfs:

$$f_{XY}(x, y) = f_X(x)f_Y(y) = \frac{x^{\alpha_1-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha_1)\beta^{\alpha_1}}\frac{y^{\alpha_2-1}e^{-\frac{y}{\beta}}}{\Gamma(\alpha_2)\beta^{\alpha_2}} = \frac{x^{\alpha_1-1}y^{\alpha_2-1}e^{-\frac{x+y}{\beta}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}}.$$

Now follow the two steps above:

1. Solve the equations $u = x + y$ and $V = \frac{x}{x+y}$ in terms of x and y. We get: $x = uv$ and $y = u(1-v)$.

2. Compute the Jacobian: $\mathbf{J} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \end{vmatrix} = -\frac{1}{x+y} = -\frac{1}{u}$.

Finally to find the joint pdf of U, V use $x = uv$ and $y = u(1-v)$ in the joint pdf of X, Y: $\mathrm{f}_{UV}(u,v) = \frac{(uv)^{\alpha_1-1}[u(1-v)]^{\alpha_2-1}e^{-\frac{u}{\beta}}u}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}}$, multiply by $\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$ and rearrange to get :

$$f_{UV}(u,v) = \frac{u^{\alpha_1+\alpha_2-1}e^{-\frac{u}{\beta}}}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} \times \frac{v^{\alpha_1-1}(1-v)^{\alpha_2-1}\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}.$$

Therefore,

$$f_{UV}(u,v) = \frac{u^{\alpha_1+\alpha_2-1}e^{-\frac{u}{\beta}}}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} \times \frac{v^{\alpha_1-1}(1-v)^{\alpha_2-1}}{B(\alpha_1,\alpha_2)},$$

where, $B(\alpha_1, \alpha_2) = \int_0^1 v^{\alpha_1-1}(1-v)^{\alpha_2-1}dv = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$ is the Beta function.

We observe that

a. U, V are independent.

b. $U \sim \Gamma(\alpha_1 + \alpha_2, \beta)$.

c. $V \sim \mathrm{Beta}(\alpha_1, \alpha_2)$.