Week 8 - practice problems solutions

(a). FROM LECTURE NOTES ON CONSTRAINED LEAST SQUARES:

$$\hat{\underline{\beta}}_C = \hat{\underline{\beta}} - (\underline{X}'\underline{X})^{-1}\underline{C}'\left[\underline{C}(\underline{X}'\underline{X})^{-1}\underline{C}'\right]^{-1}(\underline{C}\hat{\underline{\beta}}-\underline{\gamma})$$

$$\hookrightarrow \hat{\underline{\beta}}_C - \hat{\underline{\beta}} = -(\underline{X}'\underline{X})^{-1}\underline{C}'\left(\underline{C}(\underline{X}'\underline{X})^{-1}\underline{C}'\right)^{-1}(\underline{C}\hat{\underline{\beta}}-\underline{\gamma})$$

MULTIPLY BOTH SIDES BY $\underline{X}$:

$$\underline{X}(\hat{\underline{\beta}}_C - \hat{\underline{\beta}}) = -\underline{X}(\underline{X}'\underline{X})^{-1}\underline{C}'\left(\underline{C}(\underline{X}'\underline{X})^{-1}\underline{C}'\right)^{-1}(\underline{C}\hat{\underline{\beta}}-\underline{\gamma})$$

$$\therefore (\hat{\underline{\beta}}_C - \hat{\underline{\beta}})'\underline{X}'\underline{X}(\hat{\underline{\beta}}_C - \hat{\underline{\beta}})$$

$$= (\underline{C}\hat{\underline{\beta}}-\underline{\gamma})'\left(\underline{C}(\underline{X}'\underline{X})^{-1}\underline{C}'\right)^{-1}\underline{C}(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{X}(\underline{X}'\underline{X})^{-1}\underline{C}'\left(\underline{C}(\underline{X}'\underline{X})^{-1}\underline{C}'\right)^{-1}(\underline{C}\hat{\underline{\beta}}-\underline{\gamma})$$

$$= (\underline{C}\hat{\underline{\beta}}-\underline{\gamma})'\left(\underline{C}(\underline{X}'\underline{X})^{-1}\underline{C}'\right)^{-1}(\underline{C}\hat{\underline{\beta}}-\underline{\gamma})$$

WE HAVE SEEN THAT

$$\underline{e}_c'\underline{e}_c = \underline{e}'\underline{e} + (\underline{C}\hat{\underline{\beta}}-\underline{\gamma})'\left(\underline{C}(\underline{X}'\underline{X})^{-1}\underline{C}'\right)^{-1}(\underline{C}\hat{\underline{\beta}}-\underline{\gamma})$$

OR $$\underline{e}_c'\underline{e}_c = \underline{e}'\underline{e} + (\hat{\underline{\beta}}_C - \hat{\underline{\beta}})'\underline{X}'\underline{X}(\hat{\underline{\beta}}_C - \hat{\underline{\beta}})$$

(b)

$$\operatorname{Cov}\left[\hat{\beta},\ \hat{\beta}_C\right]$$

$$= \operatorname{Cov}\left(\hat{\beta},\ \hat{\beta} - (X'X)^{-1}C'\left(C(X'X)^{-1}C'\right)^{-1}\left(C\hat{\beta}-\gamma\right)\right)$$

$$= \operatorname{Cov}\left(\hat{\beta},\ \left\{I - (X'X)^{-1}C'\left(C(X'X)^{-1}C'\right)^{-1}C\right\}\hat{\beta} + \text{constant}\right)$$

$$= \sigma^2 (X'X)^{-1}\left(I - C'\left(C(X'X)^{-1}C'\right)^{-1}C(X'X)^{-1}\right)$$

WE USE HERE THE GENERAL RESULT:

$$\operatorname{Cov}\left(A\hat{\beta},\ B\hat{\beta}\right) = A \operatorname{var}(\hat{\beta})\, B'.$$

(c)

$$SSR = \sum (\hat{y}_i - \bar{y})^2$$

$$= \sum \hat{y}_i^2 - n\bar{y}^2$$

$$= \hat{\underline{y}}'\hat{\underline{y}} - n\bar{y}^2 = \hat{\underline{\beta}}'\underline{X}'\underline{X}\hat{\underline{\beta}} - n\bar{y}^2$$

BUT $\hat{\underline{y}} = \underline{X}\hat{\underline{\beta}}$

EXPECTED VALUE:

$$E\,SSR = E\left(\hat{\underline{\beta}}'\underline{X}'\underline{X}\hat{\underline{\beta}} - n\bar{y}^2\right)$$

$$= E\,\mathrm{tr}\,\hat{\underline{\beta}}'\underline{X}'\underline{X}\hat{\underline{\beta}} - n\,E\bar{y}^2$$

$$= \mathrm{tr}\,\underline{X}'\underline{X}\,E(\hat{\underline{\beta}}\hat{\underline{\beta}}') - n\left(\mathrm{var}(\bar{y}) + (E\bar{y})^2\right)$$

$$\underline{y} = \underline{X}\underline{\beta} + \underline{\varepsilon} \rightarrow \bar{Y} = \frac{\underline{1}'(\underline{X}\underline{\beta}+\underline{\varepsilon})}{n} = \frac{\underline{1}'\underline{X}\underline{\beta}}{n} + \frac{\underline{1}'\underline{\varepsilon}}{n}$$

$$= \operatorname{tr} X'X\left(\sigma^2 (X'X)^{-1} + X\beta\beta'X'\right)$$

$$- n\left[\frac{\sigma^2}{n} + \left(\frac{1'X\beta}{n}\right)^2\right]$$

$$= \sigma^2(k+1) + \beta'X'X'XX\beta$$

$$- \sigma^2 - \frac{(1'X\beta)^2}{n}$$

$$= k\sigma^2 + (X\beta)'(X\beta) - \frac{(1'X\beta)^2}{n}.$$

(d)

$$Y = X\beta + \varepsilon$$

$$C\beta = \gamma \qquad C_1\beta_1 + C_2\beta_2 = \gamma$$

$$\beta_2 = C_2^{-1}(\gamma - C_1\beta_1)$$

$$Y = X_1\beta_1 + X_2\beta_2 + \varepsilon$$

$$Y = X_1\beta_1 + X_2C_2^{-1}(\gamma - C_1\beta_1) + \varepsilon$$

$$Y - X_2C_2^{-1}\gamma = (X_1 - X_2C_2^{-1}C_1)\beta_1 + \varepsilon$$

$$Y_r = X_{1r}\beta_1 + \varepsilon$$

$$\hat{\beta}_{1c} = (X_{1r}'X_{1r})^{-1}X_{1r}'Y$$

Then $\hat{\underset{\sim}{\beta}}_{2c} = \underset{\sim}{C}_2^{-1}\left(\underset{\sim}{y} - \underset{\sim}{C}_1 \hat{\underset{\sim}{\beta}}_{1c}\right)$

$$\hat{\underset{\sim}{\beta}}_c = \begin{pmatrix} \hat{\underset{\sim}{\beta}}_{1c} \\ \hat{\underset{\sim}{\beta}}_{2c} \end{pmatrix}$$

$$\mathrm{Cov}\left(\hat{\underset{\sim}{\beta}}_{1c}, \hat{\underset{\sim}{\beta}}_{2c}\right) = \mathrm{Cov}\left(\hat{\underset{\sim}{\beta}}_{1c}, \underset{\sim}{C}_2^{-1}\left(\underset{\sim}{y} - \underset{\sim}{C}_1 \hat{\underset{\sim}{\beta}}_{1c}\right)\right)$$

$$= \mathrm{Cov}\left(\hat{\underset{\sim}{\beta}}_{1c}, \underbrace{\underset{\sim}{C}_2^{-1}\underset{\sim}{y}}_{\text{CONSTANT}} - \underset{\sim}{C}_2^{-1}\underset{\sim}{C}_1 \hat{\underset{\sim}{\beta}}_{1c}\right)$$

$$= -\mathrm{Var}\left(\hat{\underset{\sim}{\beta}}_{1c}\right)\underset{\sim}{C}_1' \underset{\sim}{C}_2^{-1\prime}$$

$$= -\sigma^2\left(\underset{\sim}{X}_{1r}' \underset{\sim}{X}_{1r}\right)^{-1} \underset{\sim}{C}_1' \underset{\sim}{C}_2^{-1\prime}$$

(e)

$$\underset{\sim}{Y}'A\underset{\sim}{Y} = \frac{1}{n-k+1}\underset{\sim}{Y}'(I-H)\underset{\sim}{Y}\ \frac{n-k-1}{n-k-1}$$

$$= \frac{n-k-1}{n-k+1} S_e^2 = (\text{SAY } \hat{\theta})$$

$$E(\hat{\theta}-\sigma^2)^2 = MSE(\hat{\theta})$$

$$= \text{var}(\hat{\theta}) + B^2$$

$$= \frac{(n-k-1)^2}{(n-k+1)^2}\frac{2\sigma^4}{n-k-1} + \left(\frac{n-k-1}{n-k+1}\sigma^2-\sigma^2\right)^2$$

$$n-k-1-n+k-1$$

$$= \frac{2\sigma^4(n-k-1)}{(n-k+1)^2} + \frac{4\sigma^4}{(n-k+1)^2}$$

$$= \frac{2\sigma^4(n-k-1+2)}{(n-k+1)^2} = \frac{2\sigma^4}{n-k+1}$$

$$< \frac{2\sigma^4}{n-k-1} = \text{var}(S_e^2)$$

(f)

$\underset{\sim}{Y}$ on $\underset{\sim}{X_1}$ $\underset{\sim}{X_2}$

$$\begin{pmatrix} \underset{\sim}{X_1}'\underset{\sim}{X_1} & \underset{\sim}{X_1}'\underset{\sim}{X_2} \\ \underset{\sim}{X_2}'\underset{\sim}{X_1} & \underset{\sim}{X_2}'\underset{\sim}{X_2} \end{pmatrix} \begin{pmatrix} \hat{\underset{\sim}{\beta}}_{1.2} \\ \hat{\underset{\sim}{\beta}}_{2.1} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{X_1}'\underset{\sim}{Y} \\ \underset{\sim}{X_2}'\underset{\sim}{Y} \end{pmatrix}$$

$$\hat{\underset{\sim}{\beta}}_{1.2} = \left(\underset{\sim}{X_1}^{*\prime}\underset{\sim}{X_1}^{*}\right)^{-1}\underset{\sim}{X_1}^{*\prime}\underset{\sim}{Y}$$

$$\hat{\underset{\sim}{\beta}}_{2.1} = \left(\underset{\sim}{X_2}^{*\prime}\underset{\sim}{X_2}^{*}\right)^{-1}\underset{\sim}{X_2}^{*\prime}\underset{\sim}{Y}$$

$\underset{\sim}{Y}$ on $\underset{\sim}{X_1}$ and $\underset{\sim}{X_2}^{*}$

$$\begin{pmatrix} \underset{\sim}{X_1}'\underset{\sim}{X_1} & \underset{\sim}{X_1}'\underset{\sim}{X_2}^{*} \\ \underset{\sim}{X_2}^{*\prime}\underset{\sim}{X_1} & \underset{\sim}{X_2}^{*\prime}\underset{\sim}{X_2}^{*} \end{pmatrix} \begin{pmatrix} \hat{\underset{\sim}{\beta}}_{1.2} \\ \hat{\underset{\sim}{\beta}}_{2.1} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{X_1}'\underset{\sim}{Y} \\ \underset{\sim}{X_2}'\underset{\sim}{Y} \end{pmatrix}$$

However, $\underset{\sim}{X_1}'\underset{\sim}{X_2}^{*} = 0$ and $\underset{\sim}{X_2}^{*\prime}\underset{\sim}{X_1} = 0$

$$\hat{\underline{\beta}}_{1\cdot 2} = (\underline{X}_1'\underline{X}_1)^{-1}\underline{X}_1'\underline{Y}$$

SAME AS IN REGRESSION $\underline{Y}$ ON $\underline{X}_1$ ALONE

$$\hat{\underline{\beta}}_{2\cdot 1} = (\underline{X}_2^{o\prime}\underline{X}_2^{o})^{-1}\underline{X}_2^{o\prime}\underline{Y}$$

SAME AS IN REGRESSION OF $\underline{Y}$ ON $\underline{X}_1$ AND $\underline{X}_2$

(g)

$$\underset{\sim}{Y}_r = \underset{\sim}{X}_{1r} \underset{\sim}{\beta}_1 + \underset{\sim}{\varepsilon}$$

$$\hat{\underset{\sim}{\beta}}_{1c} = (\underset{\sim}{X}_{1r}' \underset{\sim}{X}_{1r})^{-1} \underset{\sim}{X}_{1r}' \underset{\sim}{Y}_r$$

$$\underset{\sim}{Y}_r \sim N(\underset{\sim}{X}_{1r} \underset{\sim}{\beta}_1, \sigma^2 \underset{\sim}{I})$$

$$\hat{\underset{\sim}{\beta}}_{1c} \sim N(\underset{\sim}{\beta}_1, \sigma^2 (\underset{\sim}{X}_{1r}' \underset{\sim}{X}_{1r})^{-1})$$

Let $\underset{\sim}{V} = (\underset{\sim}{X}_{1r}' \underset{\sim}{X}_{1r})^{1/2} (\hat{\underset{\sim}{\beta}}_{1c} - \underset{\sim}{\beta}_1)$

Then $\underset{\sim}{V} \sim N(\underset{\sim}{0}, \sigma^2 \underset{\sim}{I}_p)$

And $\dfrac{\underset{\sim}{V}'\underset{\sim}{V}}{\sigma^2} \sim \chi^2_p$

$$\therefore \frac{(\hat{\underset{\sim}{\beta}}_{1c} - \underset{\sim}{\beta}_1)' \underset{\sim}{X}_{1r}' \underset{\sim}{X}_{1r} (\hat{\underset{\sim}{\beta}}_{1c} - \underset{\sim}{\beta}_1)}{\sigma^2} \sim \chi^2_p.$$

(h)

$$Y_r = X_r \beta_1 + \varepsilon$$

THE COLUMN OF ONES HAS BEEN TRANSFORMED. LET'S DENOTE IT WITH $\underset{\sim}{1}_T$

- REGRESS THE OTHER TRANSFORMED PREDICTORS ON $\underset{\sim}{1}_T$ AND OBTAINED THE RESIDUALS X_r^*

- REGRESS $\underset{\sim}{Y}_r$ ON $\underset{\sim}{1}_T$ AND OBTAINED THE RESIDUALS $\underset{\sim}{Y}_r^*$

FINALLY, REGRESS $\underset{\sim}{Y}_r^*$ ON $\underset{\sim}{X}_r^*$ TO FIND ESTIMATES OF THE SLOPES OF THE TRANSFORMED PREDICTORS WHICH ARE $\hat{\underset{\sim}{\beta}}_{1C}$ WITHOUT THE FIRST ELEMENT.

(i)

$$\sum (y_i - \bar{y})(\hat{y}_i - \bar{y})$$

$\pm \hat{y}_i$

$$= \sum (y_i - \hat{y}_i + \hat{y}_i - \bar{y})(\hat{y}_i - \bar{y})$$

$$= \sum \underbrace{e_i (\hat{y}_i - \bar{y})}_{0} + \sum (\hat{y}_i - \bar{y})^2$$

$$= \sum (\hat{y}_i - \bar{y})^2$$

Also $\sum (y_i - \bar{y})(\hat{y}_i - \bar{y})$

$$= \left(\underline{Y} - \tfrac{1}{n}\underline{1}\,\underline{1}'\underline{Y}\right)'\left(\underline{\hat{Y}} - \tfrac{1}{n}\underline{1}\,\underline{1}'\underline{Y}\right)$$

$$= \underline{Y}'\left(\underline{I} - \tfrac{1}{n}\underline{1}\,\underline{1}'\right)'\left(\underline{H} - \tfrac{1}{n}\underline{1}\,\underline{1}'\right)\underline{Y}$$

(j)

$$H = P \Lambda P'$$

THERE ARE $k+1$ EIGENVALUES EQUAL TO 1.

LET $P = \begin{pmatrix} E & F \end{pmatrix}$

$$H = \begin{pmatrix} E & F \end{pmatrix} \begin{pmatrix} I_{k+1} & 0 \\ 0 & 0_{n-k-1} \end{pmatrix} \begin{pmatrix} E' \\ F' \end{pmatrix}$$

$$= \begin{pmatrix} E & 0 \end{pmatrix} \begin{pmatrix} E' \\ F' \end{pmatrix} = EE'.$$

ALSO, $P'P = I$

$$\begin{pmatrix} E' \\ F' \end{pmatrix} \begin{pmatrix} E & F \end{pmatrix} = \begin{pmatrix} I_{k+1} & 0 \\ 0 & I_{n-k-1} \end{pmatrix}$$

$$\begin{pmatrix} E'E & E'F \\ F'E & F'F \end{pmatrix} = \begin{pmatrix} I_{k+1} & 0 \\ 0 & I_{n-k-1} \end{pmatrix}$$

$$\therefore \quad E'E = I_{k+1}$$