QUIZ 1 SOLUTIONS

(a). $E\hat{\beta}_1 = E\left(\frac{\sum(x_i-\bar{x})y_i}{\sum(x_i-\bar{x})^2}\right)$

$= \frac{1}{\sum(x_i-\bar{x})^2}\sum(x_i-\bar{x})Ey_i$ WHERE $Ey_i = \beta_0+\beta_1 x_i$

$= \frac{\sum(x_i-\bar{x})(\beta_0+\beta_1 x_i)}{\sum(x_i-\bar{x})^2}$

$= \frac{\beta_0\sum(x_i-\bar{x})+\beta_1\sum(x_i-\bar{x})x_i}{\sum(x_i-\bar{x})^2} = \beta_1$

NOTE: $\sum(x_i-\bar{x})=0$

AND $\sum(x_i-\bar{x})x_i = \sum(x_i-\bar{x})^2$

$VAR(\hat{\beta}_1) = VAR\left(\frac{\sum(x_i-\bar{x})y_i}{\sum(x_i-\bar{x})^2}\right) = \frac{1}{\left(\sum(x_i-\bar{x})^2\right)^2}\sum(x_i-\bar{x})^2 VAR(y_i)$

$= \sigma^2\frac{\sum(x_i-\bar{x})^2}{\left(\sum(x_i-\bar{x})^2\right)^2} = \frac{\sigma^2}{\sum(x_i-\bar{x})^2}$

(b) $E(\hat{\beta}_0) = E(\bar{y} - \hat{\beta}_1 \bar{x}) = E\bar{y} - \bar{x} E\hat{\beta}_1$

NOTE: $\bar{y} = \dfrac{\sum y_i}{n} = \dfrac{\sum(\beta_0 + \beta_1 x_i + \varepsilon_i)}{n}$

$$= \beta_0 + \beta_1 \bar{x} + \frac{\sum \varepsilon_i}{n}$$

$$\therefore \ E\bar{y} = n\beta_0 + \beta_1 \bar{x}$$

THEREFORE $E\hat{\beta}_0 = \beta_0 + \beta_1 \bar{x} - \beta_1 \bar{x} = \beta_0$

$$VAR(\hat{\beta}_0) = VAR(\bar{y} - \hat{\beta}_1 \bar{x})$$

$$= VAR(\bar{y}) + \bar{x}^2 VAR(\hat{\beta}_1) - 2\bar{x} \underbrace{Cov(\bar{y}, \hat{\beta}_1)}_{0}$$

$$= \frac{\sigma^2}{n} + \frac{\bar{x}^2 \sigma^2}{\sum (x_i - \bar{x})^2} = \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{\sum (x_i - \bar{x})^2} \right)$$

(c). $\sum y_i^2 = \sum (y_i \pm \hat{y}_i)^2 = \sum (y_i - \hat{y}_i + \hat{y}_i)^2$

$$= \sum (e_i + \hat{y}_i)^2 = \sum e_i^2 + \sum \hat{y}_i^2 + 2 \underbrace{\sum e_i \hat{y}_i}_{0}$$

FROM HOMEWORK 1
EXERCISE 2(c)

(d). $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \qquad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

$\hat{y}_i = \bar{y} + \hat{\beta}_1 (x_i - \bar{x})$

$\therefore \sum (\hat{y}_i - \bar{y})^2 = \sum \left(\bar{y} + \hat{\beta}_1 (x_i - \bar{x}) - \bar{y} \right)^2$

$= \hat{\beta}_1^2 \sum (x_i - \bar{x})^2 = \hat{\beta}_1^2 (n-1) S_x^2$

(e). SINCE $E\varepsilon_i = 0$ IT FOLLOWS THAT

$E y_i = \beta_0 + \beta_1 x_i$

$\therefore$ YES, THEY ARE STILL UNBIASED

(f). $c y_i = \beta_0^* + \beta_1^* d x_i + \varepsilon_i$

$y_i^* = \beta_0^* + \beta_1^* x_i^* + \varepsilon_i$

$\hat{\beta}_1^* = \dfrac{\sum (x_i^* - \bar{x}^*) y_i^*}{\sum (x_i^* - \bar{x}^*)^2} = \dfrac{cd \sum (x_i - \bar{x}) y_i}{d^2 \sum (x_i - \bar{x})^2} = \dfrac{c}{d} \hat{\beta}_1.$

$\hat{\beta}_0^* = \bar{y}^* - \hat{\beta}_1^* \bar{x}^* = c\bar{y} - \dfrac{c}{d} \hat{\beta}_1 d \bar{x}$

$\hat{\beta}_0^* = c\bar{y} - c \hat{\beta}_1 \bar{x} = c(\bar{y} - \hat{\beta}_1 \bar{x}) = c\hat{\beta}_0$

(g). $f(y_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y_i - \delta_0 - \delta_1(x_i - \bar{x}))^2}$

$= (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2}(y_i - \delta_0 - \delta_1(x_i - \bar{x}))^2}$

$L = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(y_i - \delta_0 - \delta_1(x_i - \bar{x}))^2}$

$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum(y_i - \delta_0 - \delta_1(x_i - \bar{x}))^2$

$\frac{\partial \ln L}{\partial \delta_0} = +\frac{2}{2\sigma^2}\sum(y_i - \delta_0 - \delta_1(x_i - \bar{x})) = 0$

$\sum y_i - n\hat{\delta}_0 - \hat{\delta}_1 \underbrace{\sum(x_i - \bar{x})}_{0} = 0$

$\therefore \hat{\delta}_0 = \bar{y}$

$\frac{\partial \ln L}{\partial \delta_1} = +\frac{2}{2\sigma^2}\sum(y_i - \hat{\delta}_0 - \hat{\delta}_1(x_i - \bar{x}))(x_i - \bar{x}) = 0$

$\sum(y_i - \bar{y} - \hat{\delta}_1(x_i - \bar{x}))(x_i - \bar{x}) = 0$

$\sum(y_i - \bar{y})(x_i - \bar{x}) = \hat{\delta}_1\sum(x_i - \bar{x})^2$

$\therefore \hat{\delta}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$ SAME AS $\hat{\beta}_1$ OF THE MODEL

$y_i = \beta_0 + \beta_1 X_i + \epsilon_i$

FINALLY, $\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum\left(y_i - \hat{\delta}_0 - \hat{\delta}_1(x_i-\bar{x})\right)^2 = 0$

$$\hat{\sigma}^2 = \frac{\sum\left(y_i-\hat{\delta}_0-\hat{\delta}_1(x_i-\bar{x})\right)^2}{n}$$

$$= \frac{\sum\left(y_i-\bar{y}-\hat{\beta}_1(x_i-\bar{x})\right)^2}{n} = \frac{\sum e_i^2}{n}$$

SAME AS THE MLE OF σ^2 USING THE MODEL $y_i=\beta_0+\beta_1 x_i+\varepsilon_i$

(h). $\hat{\delta}_0 = \bar{y} \sim N\left(\delta_0, \frac{\sigma}{\sqrt{n}}\right)$

NOTE: $\bar{y} = \frac{\sum y_i}{n} = \frac{\sum\left(\delta_0+\delta_1(x_i-\bar{x})+\varepsilon_i\right)}{n} = \delta_0 + \frac{\sum \varepsilon_i}{n}$

$$\hat{\delta}_1 = \hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma}{\sqrt{\sum(x_i-\bar{x})^2}}\right)$$

(i). $E\hat{\sigma}^2 = \frac{1}{n} E \frac{\sum_{i=1}^{n} e_i^2}{n} = \frac{1}{n}\sum_{i=1}^{n}\left[\operatorname{var}(e_i) + (Ee_i)^2\right]$

NOTE: $Ee_i = E(y_i - \hat{y}_i) = Ey_i - E\hat{y}_i$

$= \beta_0 + \beta_1 x_i - (\beta_0 + \beta_1 x_i) = 0$

$\therefore$ YES, $E\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} \operatorname{var}(e_i)$

(j). $E\left[\sum \varepsilon_i\right] = 0$ BECAUSE OF GAUSS-MARKOV CONDITIONS. WE DON'T KNOW WHAT $\sum \varepsilon_i$ IS.

$\operatorname{var}(\bar{\varepsilon}) = \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n} \varepsilon_i\right) = \frac{\sigma^2}{n}$

$\bar{\varepsilon} \sim N\left(0, \frac{\sigma}{\sqrt{n}}\right)$