QUIZ 2 SOLUTIONS

(a). $\sum \left[\frac{Y_i - \beta_0 - \beta_1 X_i}{\sigma} \right]^2 \sim \chi^2_n$

(b). $\sum \frac{[Y_i - \beta_0 - \beta_1 X_i]^2}{\sigma^2} = \sum \frac{(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i + (\hat{\beta}_0 - \beta_0) + (\hat{\beta}_1 - \beta_1) X_i)^2}{\sigma^2}$

$= \sum \frac{[e_i + (\hat{\beta}_0 - \beta_0) + (\hat{\beta}_1 - \beta_1) X_i]^2}{\sigma^2}$

$= \frac{\sum e_i^2}{\sigma^2} + \frac{n(\hat{\beta}_0 - \beta_0)^2}{\sigma^2} + \frac{(\hat{\beta}_1 - \beta_1)^2 \sum X_i^2}{\sigma^2}$

$+ \frac{2(\hat{\beta}_0 - \beta_0)\sum e_i}{\sigma^2} + \frac{2(\hat{\beta}_1 - \beta_1)\sum e_i X_i}{\sigma^2} + \frac{2(\hat{\beta}_0 - \beta_0)(\hat{\beta}_1 - \beta_1)\sum X_i}{\sigma^2}$

NOTE: $\sum e_i = 0$ AND $\sum e_i X_i = 0$

$\therefore \sum \frac{(Y_i - \beta_0 - \beta_1 X_i)^2}{\sigma^2} = \frac{\sum e_i^2}{\sigma^2} + \frac{n(\hat{\beta}_0 - \beta_0)^2}{\sigma^2} + \frac{(\hat{\beta}_1 - \beta_1)^2 \sum X_i^2}{\sigma^2} + \frac{2(\hat{\beta}_0 - \beta_0)(\hat{\beta}_1 - \beta_1)\sum X_i}{\sigma^2}$

$Q = \hat{\beta}_0 + \hat{\beta}_1 \bar{X} = \bar{Y}$

$\mathrm{VAR}(Q) = \sigma^2/n$

(c) $\frac{(\hat{\beta}_1 - \beta_1)^2}{\mathrm{VAR}(\hat{\beta}_1)} + \frac{(Q - (\beta_0 + \beta_1 \bar{X}))^2}{\mathrm{VAR}(Q)}$

$= \frac{(\hat{\beta}_1 - \beta_1)^2 \sum (X_i - \bar{X})^2}{\sigma^2} + \frac{((\hat{\beta}_0 - \beta_0) + (\hat{\beta}_1 - \beta_1)\bar{X})^2}{\sigma^2/n}$

$$= \frac{(\hat{\beta}_1-\beta_1)^2\left[\sum x_i^2 - n\bar{x}^2\right]}{\sigma^2} + \frac{n(\hat{\beta}_0-\beta_0)^2}{\sigma^2} + \frac{(\hat{\beta}_1-\beta_1)^2 n\bar{x}^2}{\sigma^2} + \frac{2(\hat{\beta}_0-\beta_0)(\hat{\beta}_1-\beta_1)n\bar{x}}{\sigma^2}$$

$$= \frac{(\hat{\beta}_1-\beta_1)^2\sum x_i^2}{\sigma^2} - \frac{(\hat{\beta}_1-\beta_1)^2 n\bar{x}^2}{\sigma^2} + \frac{n(\hat{\beta}_0-\beta_0)^2}{\sigma^2} + \frac{(\hat{\beta}_1-\beta_1)^2 n\bar{x}^2}{\sigma^2} + \frac{2(\hat{\beta}_0-\beta_0)(\hat{\beta}_1-\beta_1)\sum x_i}{\sigma^2}$$

$$= \frac{n(\hat{\beta}_0-\beta_0)^2}{\sigma^2} + \frac{(\hat{\beta}_1-\beta_1)^2\sum x_i^2}{\sigma^2} + \frac{2(\hat{\beta}_0-\beta_0)(\hat{\beta}_1-\beta_1)\sum x_i}{\sigma^2}$$

(d) YES, THEY ARE INDEPENDENT.

$$\operatorname{Cov}(e_i, \hat{\beta}_1) = \operatorname{Cov}(Y_i - \bar{Y} - \hat{\beta}_1(x_i-\bar{x}), \hat{\beta}_1) = 0$$

$\therefore$ Q_1, Q_2 ARE INDEPENDENT

$$\operatorname{Cov}(e_i, \bar{Y}) = \operatorname{Cov}(Y_i - \bar{Y} - \hat{\beta}_1(x_i-\bar{x}), \bar{Y}) = 0$$

$\therefore$ Q_1, Q_3 ARE INDEPENDENT

$$\operatorname{Cov}(\hat{\beta}_1, \bar{Y}) = 0$$ $\therefore$ Q_2, Q_3 ARE INDEPENDENT.

(e) $\left.\begin{array}{l} Q_2 \sim \chi^2_1 \\ Q_3 \sim \chi^2_1 \end{array}\right\}$ $Q_2 + Q_3 \sim \chi^2_2$

$$M_Q(t) = M_{Q_1}(t) \cdot M_{Q_2}(t) \cdot M_{Q_3}(t)$$

$$M_{Q_1}(t) = (1-2t)^{-\frac{n-2}{2}}$$

$$\therefore Q_1 = \frac{\sum e_i^2}{\sigma^2} = \frac{(n-2)S_e^2}{\sigma^2} \sim \chi^2_{n-2}$$

(f) $\mathrm{cov}(\hat{\beta}_0, e_i) = \mathrm{cov}(\bar{Y} - \hat{\beta}_1 \bar{X},\ Y_i - \bar{Y} - \hat{\beta}_1 (X_i - \bar{X}))$

$= \mathrm{cov}(\bar{Y}, Y_i) - \mathrm{var}(\bar{Y}, \bar{Y}) - (X_i - \bar{X}) \mathrm{cov}(\bar{Y}, \hat{\beta}_1)$

$- \bar{X} \mathrm{cov}(\hat{\beta}_1, Y_i) + \bar{X} \mathrm{cov}(\hat{\beta}_1, \bar{Y}) + \bar{X}(X_i - \bar{X}) \mathrm{var}(\hat{\beta}_1, \hat{\beta}_1)$

$$= \frac{\sigma^2}{n} - \frac{\sigma^2}{n} - 0 - \bar{X} \frac{(X_i - \bar{X})\sigma^2}{\Sigma (X_i - \bar{X})^2} + 0 + \frac{\bar{X}(X_i - \bar{X})\sigma^2}{\Sigma (X_i - \bar{X})^2} = 0$$

YES, THEY ARE INDEPENDENT

(g) $E(\hat{\beta}_1 - \hat{\beta}_0) = E\hat{\beta}_1 - E\hat{\beta}_0 = \beta_1 - \beta_0.$

$\mathrm{var}(\hat{\beta}_1 - \hat{\beta}_0) = \mathrm{var}(\hat{\beta}_1) + \mathrm{var}(\hat{\beta}_0) - 2\,\mathrm{cov}(\hat{\beta}_1, \hat{\beta}_0)$

$$= \frac{\sigma^2}{\Sigma (X_i - \bar{X})^2} + \sigma^2 \left(\frac{1}{n} + \frac{\bar{X}^2}{\Sigma (X_i - \bar{X})^2} \right) + 2 \bar{X} \frac{\sigma^2}{\Sigma (X_i - \bar{X})^2}$$

$$= \sigma^2 \left[\frac{1}{n} + \frac{(\bar{X} + 1)^2}{\Sigma (X_i - \bar{X})^2} \right)$$

(h). $\mathrm{cov}(\hat{\beta}_1 - \hat{\beta}_0, e_i) = \mathrm{cov}(\hat{\beta}_1, e_i) - \mathrm{cov}(\hat{\beta}_0, e_i)$

$= \mathrm{cov}(\hat{\beta}_1,\ Y_i - \bar{Y} - \hat{\beta}_1 (X_i - \bar{X})) - \mathrm{cov}(\bar{Y} - \hat{\beta}_1 \bar{X},\ Y_i - \bar{Y} - \hat{\beta}_1 (X_i - \bar{X}))$

$$= \frac{\sigma^2 (X_i - \bar{X})}{\Sigma (X_i - \bar{X})^2} - 0 - \frac{\sigma^2 (X_i - \bar{X})}{\Sigma (X_i - \bar{X})^2} - \left[\frac{\sigma^2}{n} - \frac{\sigma^2}{n} - 0 - \frac{\bar{X}(X_i - \bar{X})\sigma^2}{\Sigma (X_i - \bar{X})^2} + 0 + \frac{\bar{X}(X_i - \bar{X})\sigma^2}{\Sigma (X_i - \bar{X})^2} \right]$$

$= 0.$

(i), $\hat{\beta}_1 = \dfrac{\sum x_i y_i}{\sum x_i^2}$ $\quad \hat{y}_i = \hat{\beta}_1 x_i = x_i \dfrac{\sum x_j y_j}{\sum x_j^2}$

$\hat{y}_j = x_j \dfrac{\sum x_i y_i}{\sum x_i^2}$

$e_i = y_i - \hat{y}_i$

$e_j = y_j - \hat{y}_j$

$$\text{cov}(\hat{y}_i, \hat{y}_j) = \text{Cov}\left(x_i \frac{\sum x_j y_j}{\sum x_j^2}, \ x_j \frac{\sum x_i y_i}{\sum x_i^2}\right)$$

$$= \sigma^2 \frac{x_i x_j}{\sum x_i^2}.$$

$$\text{Cov}(e_i, e_j) = \text{cov}(y_i - \hat{y}_i, \ y_j - \hat{y}_j)$$

$$= \text{cov}(y_i, y_j) - \text{Cov}(y_i, \hat{y}_j) - \text{Cov}(\hat{y}_i, y_j) + \text{Cov}(\hat{y}_i, \hat{y}_j)$$

$$= 0 - \text{Cov}(y_i, \hat{\beta}_1 x_j) - \text{Cov}(\hat{\beta}_1 x_i, y_j) + \sigma^2 \frac{x_i x_j}{\sum x_i^2}$$

$$= 0 - \frac{\sigma^2 x_i x_j}{\sum x_i^2} - \sigma^2 \frac{x_i x_j}{\sum x_i^2} + \sigma^2 \frac{x_i x_j}{\sum x_i^2} = -\sigma^2 \frac{x_i x_j}{\sum x_i^2}$$

(j) $\text{Cov}(\hat{\beta}_1, e_i) = \text{Cov}\left(\dfrac{\sum x_i y_i}{\sum x_i^2}, \ y_i - \hat{\beta}_1 x_i\right)$

$$= \sigma^2 \frac{x_i}{\sum x_i^2} - \sigma^2 \frac{x_i}{\sum x_i^2} = 0.$$