(a) $\text{MIN } (Y-\theta)'(Y-\theta) \quad \text{s.t.} \quad \underset{\sim}{1}'\theta = 2\pi$

USING LAGRANGE MULTIPLIER

$$\text{MIN } Q = Y'Y - 2Y\theta + \theta'\theta + 2\lambda'(\underset{\sim}{1}'\theta - 2\pi)$$

$$\frac{\partial Q}{\partial \theta} = -2Y + 2\theta + 2\underset{\sim}{1}\lambda = 0$$

$$\hat{\theta} = Y - \lambda \underset{\sim}{1}$$

$$1'\hat{\theta} = 1'(Y - \lambda \underset{\sim}{1})$$

$$2\pi = 1'Y - \lambda 4 \rightarrow \lambda = \frac{1'Y - 2\pi}{4} = \bar{Y} - \frac{\pi}{2}$$

$$\therefore \hat{\theta} = \underset{\sim}{Y} - \underset{\sim}{1}\left(\bar{Y} - \frac{\pi}{2}\right) \quad \text{i.e.} \quad \boxed{\hat{\theta}_i = Y_i - \bar{Y} + \frac{\pi}{2}}$$

IT FOLLOWS THAT $SSE = \sum (Y_i - \hat{\theta}_i)^2 = 4\left(\bar{Y} - \frac{\pi}{2}\right)^2$

UNDER H_0: $\theta_1 = \theta_3 = \alpha_1$ and $\theta_2 = \theta_4 = \alpha_2 = \pi - \phi_1$

$$\hat{\phi}_1 = \frac{1}{4}(Y_1 - Y_2 + Y_3 - Y_4 + 2\pi)$$

$$\hat{\phi}_2 = \pi - \hat{\phi}_1$$

THEREFORE,

$$SSE_R = (Y_1 - \hat{\phi}_1)^2 + (Y_3 - \hat{\phi}_1)^2 + (Y_2 - \pi + \hat{\phi}_1)^2 + (Y_4 - \pi + \hat{\phi}_1)^2$$

$$F = \frac{(SSE_R - SSE_F)/2}{SSE_F/1} \qquad DF = n - 3 = 4 - 3 = 1$$

(b) $\underset{\sim}{Y} = \underset{\sim}{\theta} + \varepsilon \qquad \hat{\underset{\sim}{\theta}} = \underset{\sim}{Y}$ OLS

$$\hat{\underset{\sim}{\theta}}_c = \hat{\theta} - (X'X)^{-1} c' \left[c (X'X)^{-1} c' \right]^{-1} (c\hat{\theta} - \gamma)$$

where $c = (1, 1, 1, 1)$, $\gamma = 0$

$$= \underset{\sim}{Y} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \left[(1\,1\,1\,1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right]^{-1} \left((1\,1\,1\,1)\, \hat{\underset{\sim}{\theta}} \right)$$

$$= \underset{\sim}{Y} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{4} \sum \hat{\theta}_i$$

$\therefore \hat{\theta}_{1c} = y_1 - \bar{y}, \quad \hat{\theta}_{2c} = y_2 - \bar{y}, \quad \hat{\theta}_{3c} = y_3 - \bar{y}, \quad \hat{\theta}_{4c} = y_4 - \bar{y}$

$H_0: \theta_1 - \theta_3 = 0$

$$F = \frac{\left(1\ 0\ -1\ 0\right)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \left((1\ 0\ -1\ 0) \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right)^{-1} (1\ 0\ -1\ 0) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}}{1 \cdot S_e^2}$$

$$= \frac{(y_1 - y_3)^2}{2 S_e^2} = *$$

Finally

$$S_e^2 = \frac{\sum e_i^2}{4-3} = (y_1 - y_1 + \bar{y})^2 + (y_2 - y_2 + \bar{y})^2 + (y_3 - y_3 + \bar{y})^2 + (y_4 - y_4 + \bar{y})^2$$

$$= 4\bar{y}^2 = 4 \frac{(y_1 + y_2 + y_3 + y_4)^2}{16} = \frac{(y_1 + y_2 + y_3 + y_4)^2}{4}$$

$$* = \frac{2 (y_1 - y_3)^2}{(y_1 + y_2 + y_3 + y_4)^2}$$

(c)

Residuals for new model

$\hat{y}_s = Z_s R^{-1} Z_s' y_s$ INTERCEPT IS ZERO

WHERE $y_s = \dfrac{(I - \frac{1}{n}11')y}{a}$, $a = \sqrt{\sum (y_i - \bar{y})^2}$

So $$e^* = \left(\frac{I - \frac{1}{n}11'}{a} - \frac{Z_s R^{-1} Z_s'}{a} \right) y$$

$$SSE^* = y' \left(\frac{I - \frac{1}{n}11'}{a} - \frac{Z_s R^{-1} Z_s'}{a} \right) \left(\frac{I - \frac{1}{n}11'}{a} - \frac{Z_s R^{-1} Z_s'}{a} \right) y$$

$$= \frac{y'(I - \frac{1}{n}11')y}{a^2} - \frac{y' Z_s R^{-1} Z_s' y}{a^2} - \frac{y' Z_s R^{-1} Z_s' y}{a^2} + \frac{y' Z_s R^{-1} Z_s' y}{a^2}$$

$$= \frac{y'(I - \frac{1}{n}11')y}{a^2} - \frac{y' Z_s R^{-1} Z_s' y}{a^2}.$$

$$= \frac{y'(I - \frac{1}{n}11')y - y' \underbrace{Z_s R^{-1} Z_s'}_{Z_s (Z_s' Z_s)^{-1} Z_s'} y}{a^2} = \frac{e'e}{a^2}.$$

$Z_s (Z_s' Z_s)^{-1} Z_s'$

HAT MATRIX FOR CENTERED SCALED MODEL