WEEK 9 PRACTICE PROBLEMS SOLUTIONS

EXERCISE 1:

$$\text{MIN} \quad \text{VAR}(y_0 - \hat{y}_0) = \sigma^2 + \underline{w}'\underline{\Sigma}\underline{w} - 2\underline{c}'\underline{w}$$

$$\text{s.t.} \ \sum w_i = 1 \ \text{ OR } \ \underline{1}'\underline{w} = 1$$

$$\text{MIN} \quad Q = \sigma^2 + \underline{w}'\underline{\Sigma}\underline{w} - 2\underline{c}'\underline{w} - 2\lambda[\underline{1}'\underline{w} - 1]$$

$$\frac{\partial Q}{\partial w} = 2\underline{\Sigma}\underline{w} - 2\underline{c} - 2\underline{1}\lambda = 0$$

$$\underline{w} = \underline{\Sigma}^{-1}[\underline{c} + \lambda\underline{1}]$$ NEED TO FIND λ:

$$\underline{1}'\underline{w} = \underline{1}'\underline{\Sigma}^{-1}(\underline{c} + \lambda\underline{1}) \rightarrow \lambda = \frac{1 - \underline{1}'\underline{\Sigma}^{-1}\underline{c}}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}}$$

AND $$\underline{w} = \underline{\Sigma}^{-1}\left(\underline{c} + \frac{1 - \underline{1}'\underline{\Sigma}^{-1}\underline{c}}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}}\underline{1}\right)$$

FINALLY, $$\hat{y}_0 = \underline{w}'\underline{Y} = \left(\underline{c}' + \frac{1 - \underline{1}'\underline{\Sigma}^{-1}\underline{c}}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}}\underline{1}'\right)\underline{\Sigma}^{-1}\underline{Y}$$

$$= \underline{c}'\underline{\Sigma}^{-1}\underline{Y} + \frac{\underline{1}'\underline{\Sigma}^{-1}\underline{Y}}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}} - \frac{\underline{1}'\underline{\Sigma}^{-1}\underline{c}\,\underline{1}'\underline{\Sigma}^{-1}\underline{Y}}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}}$$

NOTE: $\hat{\mu} = \dfrac{\underline{1}'\underline{\Sigma}^{-1}\underline{Y}}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}}$

$$= \hat{\mu} + \underline{c}'\underline{\Sigma}^{-1}\underline{Y} - \underline{c}'\underline{\Sigma}^{-1}\underline{1}\,\frac{\underline{1}'\underline{\Sigma}^{-1}\underline{Y}}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}}$$ OR $$\hat{y}_0 = \hat{\mu} + \underline{c}'\underline{\Sigma}^{-1}(\underline{Y} - \hat{\mu}\underline{1})$$

EXERCISE 2 :

(a). $\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'Y$

$\hat{\beta}_2 = (X_2'X_2)^{-1}X_2'Y$

(b). $\hat{\beta} = (X'X)^{-1}X'Y$ NOTE: $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

AND $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

THEREFORE,

$$\hat{\beta} = (X_1'X_1 + X_2'X_2)^{-1}(X_1'Y_1 + X_2'Y_2)$$

(c). LET $X_1 = X_2$ THEN

$$\hat{\beta} = (2X_1'X_1)^{-1}X_1'(Y_1+Y_2) = \tfrac{1}{2}(X_1'X_1)^{-1}X_1'(Y_1+Y_2)$$

$$= \tfrac{1}{2}(X_1'X_1)^{-1}X_1'Y_1 + \tfrac{1}{2}(X_1'X_1)^{-1}X_1'Y_2$$

(d). $VAR(\hat{\beta}) = \frac{\sigma^2}{4}(X_1'X_1)^{-1}X_1'X_1(X_1'X_1)^{-1} + \frac{\sigma^2}{4}(X_1'X_1)^{-1}X_1'X_1(X_1'X_1)^{-1}$

$$= \frac{\sigma^2}{2}(X_1'X_1)^{-1}$$

(e). $W = \begin{pmatrix} \sigma^2 I_{n_1} & 0 \\ 0 & c\sigma^2 I_{n_2} \end{pmatrix}$

EXERCISE 3:

$\varepsilon_i \sim N(0,1)$ $\quad \sigma^2 = 1$ KNOWN

$y_i = \gamma_0 + \delta_1 z_{si1} + \delta_2 z_{si2} + \varepsilon_i$

$H_0: \delta_2 = 0$

$H_a: \delta_2 \neq 0$

SINCE $\sigma^2 = 1$ FIND THE DISTRIBUTION OF $\hat{\delta}_2$ AND THEN STANDARDIZE IT TO GET $N(0,1)$

$\hat{\delta}_2 \sim N(\delta_2, \sigma\sqrt{V_{22}})$ WHERE V_{22} IS ELEMENT OF R^{-1} AND $\sigma^2 = 1$

$\hat{\delta}_2 \sim N(\delta_2, \sqrt{V_{22}})$ UNDER $H_0: \delta_2 = 0$

THEREFORE, $\hat{\delta}_2 \sim N(0, \sqrt{V_{22}})$

AND THE TEST STATISTIC IS $z = \dfrac{\hat{\delta}_2 - 0}{\sqrt{V_{22}}} \sim N(0,1)$

SIMILARLY FOR TESTING $H_0: \delta_1 = 0$ FIND THE DISTRIBUTION OF $\hat{\delta}_1$ AND STANDARDIZE IT AS ABOVE TO GET $N(0,1)$.