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Statistics 110A **Instructor: Nicolas Christou**

The gamma distribution

Before we discuss the χ^2, t, F distributions which are related to the normal distribution, we need to discuss the so called gamma distribution. The gamma distribution is useful in modeling skewed distributions for variables that are not negative.

A random variable X is said to have a gamma distribution with parameters α, β if its probability density function is given by

$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)}, \quad \alpha, \beta > 0, x \geq 0.$$

$E(X) = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$.

A brief note on the *gamma* function:
The quantity $\Gamma(\alpha)$ is known as the gamma function and it is equal to:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx.$$

If $\alpha = 1$, $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$.

With integration by parts we get $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ as follows:

$$\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}dx =$$

Let, $v = x^\alpha \Rightarrow \frac{dv}{dx} = \alpha x^{\alpha-1}$
$\frac{du}{dx} = e^{-x} \Rightarrow u = -e^{-x}$
Therefore,

$$\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}dx = -e^{-x}x^\alpha|_0^\infty - \int_0^\infty -e^{-x}\alpha x^{\alpha-1}dx = \alpha\int_0^\infty x^{\alpha-1}e^{-x}dx.$$

Or, $\Gamma(\alpha+1) = \alpha\Gamma(a)$.

Similarly, using integration by parts it can be shown that,
$\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\alpha\Gamma(\alpha)$, and,
$\Gamma(\alpha+3) = (\alpha+2)(\alpha+1)\alpha\Gamma(\alpha)$.
Therefore, using this result, when α is an integer we get $\Gamma(\alpha) = (\alpha-1)!$.

Example:
$\Gamma(5) = \Gamma(4+1) = 4\times\Gamma(4) = 4\times\Gamma(3+1) = 4\times3\times\Gamma(3) = 4\times3\times\Gamma(2+1) = 4\times3\times2\times\Gamma(1+1) = 4\times3\times2\times\times1\times\Gamma(1) = 4!$.

Useful result:

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

If we set $\alpha = 1$ and $\beta = \frac{1}{\lambda}$ we get $f(x) = \lambda e^{-\lambda x}$. We see that the exponential distribution is a special case of the gamma distribution.

The gamma density for $\alpha = 1, 2, 3, 4$ and $\beta = 1$.