University of California, Los Angeles
Department of Statistics

Statistics 13 **Instructor: Nicolas Christou**

Homework 3 - Solutions

EXERCISE 1
It is given that $X \sim N(188, 24)$.

a. Done!

b. $-0.675 = \frac{Q_1 - 188}{24} \Rightarrow Q_1 = 171.8$, $0.675 = \frac{Q_3 - 188}{24} \Rightarrow Q_3 = 204.2$, $IQR = 204.2 - 171.8 = 32.4$.

c. $P(X > 220) = P(X > \frac{220-188}{24}) = P(Z > 1.33) = 1 - 0.9082 = 0.0918$.

d. $P(150 < X < 170) = P(\frac{150-188}{24} < Z < \frac{170-188}{24}) = P(-1.58 < Z < -0.75) = 0.2266 - 0.0571 = 0.1695$.

EXERCISE 2
We are given $X \sim N(70, \sigma)$. From $P(X > 79) = 0.025$ we find the corresponding z-value: $z = 1.96$. Therefore $1.96 = \frac{79-70}{\sigma} \Rightarrow \sigma = 4.59$ inches.

EXERCISE 3
We are given $X \sim N(\mu, 20)$. From $P(X > 214) = 0.05$ we find the corresponding z-value: $z = 1.645$. Therefore $1.645 = \frac{214-\mu}{20} \Rightarrow \mu = 181.1$ pounds.

EXERCISE 4

a. Let X denote the length of the tails. Then $X \sim N(\mu, \sigma)$. The mean μ, and the standard deviation σ are unknown but we will be able to find them using the information given:

   i. It is known that 5% of the tails is longer than 12 inches. From the standard normal distribution (Z) we find that $z = 1.645$. Therefore $z = \frac{x-\mu}{\sigma} \Rightarrow 1.645 = \frac{12-\mu}{\sigma}$.

   ii. It is known that 2.5% of the tails is shorter than 7 inches. From the standard normal distribution (Z) we find that $z = -1.96$. Therefore $z = \frac{x-\mu}{\sigma} \Rightarrow -1.96 = \frac{7-\mu}{\sigma}$.

   We have 2 equations with 2 unknowns:
   From (i) $\Rightarrow 12 - \mu = 1.645\sigma \Rightarrow \mu = 12 - 1.645\sigma$
   From (ii) $\Rightarrow 7 - \mu = -1.96\sigma$
   We substitute (i) into (ii) $\Rightarrow 7 - 12 + 1.645\sigma = -1.96\sigma \Rightarrow \sigma = 1.39$. Now we go back to (i) to find that $\mu = 12 - 1.645(1.39) \Rightarrow \mu = 9.71$. Therefore, the mean of the lengths of the tails is $\mu = 9.71$ inches, and the standard deviation is $\sigma = 1.39$ inches.

b. We want to compute $P(X > 11)$. We know from part (a) that $X \sim N(9.71, 1.39)$. So, $P(X > 11) = P(Z > \frac{11-9.71}{1.39}) = P(Z > 0.93) = 1 - 0.8238 = 0.1762$.

EXERCISE 5
Let X denote the height of people. Then $X \sim N(70, 3)$. To find the height for which 95% of the people is shorter, we need the value of Z which corresponds to an area of 95% (shaded area in the table). This value is $z = 1.645$. Now we can solve for X. $Z = \frac{X-\mu}{\sigma} \Rightarrow 1.645 = \frac{X-70}{3} \Rightarrow x = 70 + 1.645(3) \Rightarrow x = 74.935$ inches. But the door must have 1 foot (12 inches) clearance from this height. So the height of the doors should be $74.935 + 12 = 86.9$ or approximately 87 inches.

EXERCISE 6

a. Let X be the amount of ketchup dispensed by this machine. Then $X \sim N(28, 0.8)$. We need to find the 7_{th} percentile of this distribution. From Z table we find that $z = -1.475$. Therefore $-1.475 = \frac{x-28}{0.8} \Rightarrow x = 26.82$ ounces. So, 7% of the bottles will have weight under 26.82 ounces.

b. Let X be the amount dispensed by the machine. Then $X \sim N(25.8, \sigma)$. Now 25.8 ounces is the 11_{th} percentile. From the Z table we find that $z = -1.225$. Therefore $-1.225 = \frac{25-25.8}{\sigma} \Rightarrow \sigma = 0.65$ ounce.

EXERCISE 7

Outliers (suspected or serious) are values above $Q_3 + 1.5IQR$ or below $Q1 - 1.5IQR$. So we must find the values of Q_1, Q_3, and IQR using the Z table. The 25_{th} percentile (Q_1) and the 75_{th} percentile (Q_3) are $Q_1 = -0.675$ and $Q_3 = 0.675$ respectively. Therefore the interquartile range is $IQR = Q_3 - Q_1 = 0.675 - (-0.675) = 1.35$. As it was mentioned earlier, outliers are above $Q_3 + 1.5IQR = 0.675 + 1.5(1.35) = 2.7$ or below $Q_1 - 1.5IQR = -0.675 - 1.5(1.35) = -2.7$. So we must compute the probability $P(z > 2.7)$ or $P(z < -2.7) = 1 - 0.9965 + 0.0035 = 0.007$. Or 0.7% of the observations are outliers according to the box plot criterion. Therefore among 1000 observations $1000 \times 0.007 = 7$ are expected to be outliers.

EXERCISE 8

It is given that $n = 1000$, $p = 0.10$, and that there are 100 beds.

a. $P(X > 100) = \sum_{x=101}^{1000} \binom{1000}{x} 0.10^x 0.90^{1000-x}$.

b. $\mu = np = 1000(0.10) = 100, \sigma = \sqrt{1000(0.10)(0.90)} = 9.49$.
$P(X > 100) = P(Z > \frac{100.5-100}{9.49}) = P(Z > 0.05) = 1 - 0.5199 = 0.4801$.

b. We want the probability of part (b) to be only 2%. The z value is 2.055 (look for the 98th percentile in you z table). Therefore $2.055 = \frac{x+0.5-100}{9.49} \Rightarrow x = 119.002$. They need 120 beds.

EXERCISE 9

Let X be the weight of the chocolate chip cookies. We have $X \sim N(18, 3)$.

a. We need the z score for the 20th and 80th percentiles. You can use your z-table, but you can also use `R`:

```
> qnorm(0.20)
[1] -0.8416212
> qnorm(0.80)
[1] 0.8416212
```

Therefore the 20th and 80th percentiles of the weights are:
$x = 18 - 0.8416212(3) = 15.48$, and
$x = 18 + 0.8416212(3) = 20.52$.

b. In general the distribution of the sample mean is $\bar{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$. In our problem is $\bar{X} \sim N(18, \frac{3}{\sqrt{4}})$, or $\bar{X} \sim N(18, 1.5)$. We want a and b such that $P(a < \bar{X} < b) = 0.80$. We will need to find the z-score for the 10th and 90th percentiles. From your table, or `R` there are, -1.281552 and 1.281552. Therefore $a = 18 - 1.281552(1.5) = 16.08$ and $b = 18 + 1.281552(1.5) = 19.92$.

c. In this question we are dealing with the distribution of the sum, which follows: $T \sim N(n\mu, \sigma\sqrt{n})$. In our problem, $T \sim N(18(36), 3\sqrt{36})$, or $T \sim N(648, 18)$. Therefore, $P(T > 675) = P(Z > \frac{675-648}{18}) = P(Z > 1.5) = 1 - 0.9332 = 0.0668$.

EXERCISE 10
It is given that $X \sim N(20, 2.5)$.

a. Since Y is a linear combination of X it follows that $Y \sim N(102, 12.5)$. Therefore, $P(96 < Y < 115) = P(\frac{96-102}{12.5} < Z < \frac{115-1-2}{12.5}) = P(-0.48 < Z < 1.04) = 0.5352$.

b. First we find the probability that the height of a tree is more than 23 feet: $P(X > 23) = P(Z > \frac{23-20}{2.5}) = P(Z > 1.2) = 0.1151$. Let now W be the number of trees that have height more than 23 feet out of the 10 trees. This is a binomial distribution question: $P(W = 2) = \binom{10}{2} 0.1151^2 (0.8849)^8 = 0.2241$.

c. Since $Y \sim N(102, 12.5)$ it follows that the total cost of 10 such trees will be $T \sim N(1020, 39.53)$. The 5th percentile of this distribution is $1020 - 1.645(39.53) = 954.97$.

EXERCISE 11
Since X, Y, Z are normal random variables any linear combination X, Y, Z will also be normal as follows: $3X + 2Y - 6Z \sim N(0, 7)$. Therefore, $P(3X + 2Y - 6Z < -7) = P(Z < \frac{-7-0}{7} = P(Z < -1) = 0.1587$.

EXERCISE 12
It is given that $X_1 \sim N(184, 40)$ and $X_2 \sim N(172, 51)$. Therefore, $X_1 - X_2 \sim N(184 - 172, \sqrt{\frac{40^2}{50} + \frac{51^2}{50}})$ or $X_1 - X_2 \sim N(12, 9.17)$. It follows that the 40th percentile of the distribution of $X_1 - X_2$ is equal to $12 - 0.255(9.17) = 9.66$.

EXERCISE 13
It is given that $X_1 \sim N(184, 40)$ and $X_2 \sim N(172, 51)$.

a. $X_1 - X_2 \sim N(184 - 172, \sqrt{\frac{40^2}{16} + \frac{51^2}{16}})$ or $X_1 - X_2 \sim N(12, 16.20)$. Therefore, $P(\bar{X} - \bar{Y} > 9) = P(Z > \frac{9-12}{16.20}) = P(Z > -0.19) = 1 - 0.4247 = 0.5753$.

b. The 5th percentile of $\bar{X} - \bar{Y}$ is 12-1.645(16.20)=-14.649.

c. First we find the probability that a female Hawaiian gallinule has wing length more than 200 mm. $P(Y > 200) = P(Z > \frac{200-172}{51}) = P(Z > 0.55) = 1 - 0.7088 = 0.2912$. Let now Q be the number of female Hawaiian gallinules that have wing length more than 200 mm among the 5 selected. This is a binomial distribution question: $P(Q = 2) = \binom{5}{2} 0.2912^2 (0.7088)^3 = 0.3020$.