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Statistics 13 **Instructor: Nicolas Christou**

Permutations and Combinations

Basic principle of counting:
Suppose two experiments are to be performed. Then, if the first experiment can result in m outcomes and if for each outcome of the first experiment there are n outcomes of the second experiment, then all together there are $m \times n$ possible outcomes.
Examples:

Permutations:
How many different *ordered* arrangements of the letters A, B, C are possible? There are are 6 *permutations*: $ABC, ACB, BAC, BCA, CAB, CBA$. Or, using the basic principle of counting we can find the number of permutations as follows: $3 \times 2 \times 1 = 6$.
In general n objects can be ordered in $n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$ ways. Each arrangement it is called a *permutation*.
Example: In how many ways can 4 math books, 3 chemistry books, and 2 history books can be ordered so that books of the subject are together?

Suppose k objects are to be selected and ordered from n objects ($k < n$).
Say, $n = 4$ (A, B, C, D), and $k = 3$. Let's list all the possible permutations:

ABC	BCD	CDA	DAB
ABD	BCA	CDB	DAC
ACB	BDA	CAB	DBC
ACD	BDC	CAD	DBA
ADB	BAC	CBD	DCA
ADC	BAD	CBA	DCB

As we observe there are 24 permutations. Much easier, we can find the number of permutations using the basic principle of counting as follows: $4 \times 3 \times 2 = 24$.
In general, the number of ways that k objects can be selected and ordered from n objects are: $n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$. This can be simplified if we multiply and divide by $(n-k)!$:

$$n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)\frac{(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}.$$

Example: In how many ways can we select and order 5 cards from the 52 cards?

$$\frac{52!}{(52-5)!} = \frac{52!}{47!} = \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{47!} = 311875200.$$

Combinations:
Order does not count. For example $ABC, ACB, BAC, CAB, CBA, BCA$ are 6 permutations, but in terms of combinations they count for only one. For every 6 permutations we have one combination. In our example, if we divide 24 by 6 we get the number of combinations: $\frac{24}{6} = \frac{24}{3!} = 4$. In general: In how many ways can we select k objects from n objects:

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

A poker hand consists of 5 cards. How many poker hands are there?

$$\binom{52}{5} = \frac{52!}{(52-5)!5!} = 2598960.$$

Sometimes we refer to $\binom{n}{k}$ as the *binomial coefficient* because it appears in the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

For example,
$(x+y)^4 = \binom{4}{0}x^0y^4 + \binom{4}{1}x^1y^3 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^3y^1 + \binom{4}{4}x^4y^0 = y^4 + 4xy^3 + 6x^2y^2 + 4x^3y + x^4.$