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Statistics 13 **Instructor: Nicolas Christou**

Confidence intervals - some numerical examples

Example 1:
Suppose that the length of iron rods from a certain factory follows the normal distribution with known standard deviation $\sigma = 0.2\ m$ but unknown mean μ. Construct a 95% confidence interval for the population mean μ if a random sample of $n = 16$ of these iron rods has sample mean $\bar{x} = 6\ m$.
Answer:

$$\mu \in \bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \Rightarrow \quad 6 \pm 1.96 \frac{0.2}{\sqrt{16}} \quad \text{or} \quad 6 \pm 0.098$$

Therefore we are 95% confident that $5.9 \le \mu \le 6.1$.

Example 2:
For the example above, suppose that we want the entire width of the confidence interval to be equal to $0.05\ m$. Find the sample size n needed.
Answer:

$$n = \left(\frac{z_{\frac{\alpha}{2}} \sigma}{E} \right)^2 = \left(\frac{1.96 \times 0.2}{0.025} \right)^2 = 245.9 \approx 246.$$

Example 3:
A sample of size $n = 50$ is taken from the production of lightbulbs at a certain factory. The sample mean of the lifetime of these 50 lightbulbs is found to be $\bar{x} = 1570$ hours. Assume that the population standard deviation is $\sigma = 120$ hours.

a. Construct a 95% confidence interval for μ.
Answer:

$$1570 \pm 1.96 \frac{120}{\sqrt{50}} \Rightarrow 1570 \pm 33.3 \Rightarrow 1536.7 \le \mu \le 1603.3.$$

b. Construct a 99% confidence interval for μ.
Answer:

$$1570 \pm 2.575 \frac{120}{\sqrt{50}} \Rightarrow 1570 \pm 43.7 \Rightarrow 1526.3 \le \mu \le 1613.7.$$

c. What sample size is needed so that the length of the interval is 30 hours with 95% confidence?
Answer:

$$n = \left(\frac{z_{\frac{\alpha}{2}} \sigma}{E} \right)^2 = \left(\frac{1.96 \times 120}{15} \right)^2 = 245.9 \approx 246.$$

Example 4:
The daily production of a chemical product last week in tons was: 785, 805, 790, 793, and 802.

a. Construct a 95% confidence interval for the population mean μ.
   Answer: These data have $\bar{x} = 795$ and $sd(x) = 8.34$. Using the t distribution with 4 degrees of freedom we get:

$$\bar{x} \pm t_{\frac{\alpha}{2};n-1}\frac{s}{\sqrt{n}} \Rightarrow 795 \pm 2.776\frac{8.34}{\sqrt{5}} \Rightarrow 795 \pm 10.35 \Rightarrow 784.65 \leq \mu \leq 805.35.$$

b. What assumptions are necessary?
   Answer: The sample was selected from normal distribution.

Example 5:
A precision instrument is guaranteed to read accurately to within 2 units. A sample of 4 instrument readings on the same object yielded the measurements 353, 351, 351, and 355. Find a 90% confidence interval for the population variance. What assumptions are necessary? Does the guarantee seem reasonable?
Answer: These data have $var(x) = 3.67$. Using the χ^2 distribution we get:

$$\left[\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2};n-1}}, \quad \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2};n-1}}\right] \Rightarrow \left[\frac{(4-1)\times 3.67}{7.81}, \frac{(4-1)\times 3.67}{0.352}\right] \Rightarrow (1.41, 31.3)\,.$$

Example 6:
At a survey poll before the elections candidate A receives the support of 650 voters in a sample of 1200 voters.

a. Construct a 95% confidence interval for the population proportion p that supports candidate A.
   Answer: These data have $\hat{p} = \frac{650}{1200} = 0.54$.

$$\hat{p} \pm z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \Rightarrow 0.54 \pm 1.96\sqrt{\frac{0.54(1-0.54)}{1200}} \Rightarrow 0.54 \pm 0.028 \Rightarrow 0.512 \leq p \leq 0.568.$$

b. Find the sample size needed so that the margin of error will be ± 0.01 with confidence level 95%.
   Answer:

$$n = \frac{z^2_{\frac{\alpha}{2}}\hat{p}(1-\hat{p})}{E^2} = \frac{1.96^2(0.54)(1-0.54)}{0.01^2} = 9542.5 \approx 9543.$$

Example 7:

The *UCLA* housing office wants to estimate the mean monthly rent for studios around the campus. A random sample of size $n = 31$ studios is taken from the area around *UCLA*. The sample mean is found to be $\overline{x} = \$1300$ with sample standard deviation $s = \$150$. Assume that these data are selected from a normal distribution.

a. Construct a 90% confidence interval for the mean monthly rent of studios in the area around *UCLA*.
   Answer: Using the t distribution with 30 degrees of freedom

$$\bar{x} \pm t_{\frac{\alpha}{2};n-1}\frac{s}{\sqrt{n}} \Rightarrow 1300 \pm 1.697\frac{150}{\sqrt{31}} \Rightarrow 1300 \pm 45.7 \Rightarrow 1254.3 \leq \mu \leq 1345.7.$$

b. Construct a 99% confidence interval for the mean monthly rent of studios in the area around *UCLA*.
   Answer:

$$\bar{x} \pm t_{\frac{\alpha}{2};n-1}\frac{s}{\sqrt{n}} \Rightarrow 1300 \pm 2.750\frac{150}{\sqrt{31}} \Rightarrow 1300 \pm 74.1 \Rightarrow 1225.9 \leq \mu \leq 1374.1.$$

c. What sample size is needed so that the length of the interval is \$60 with 95% confidence? Assume $\sigma = \$150$.
   Answer:

$$n = \left(\frac{z_{\frac{\alpha}{2}}\sigma}{E}\right)^2 = \left(\frac{1.96 \times 150}{30}\right)^2 = 96.04 \approx 97.$$

Example 8:
A chemist has prepared a product designed to kill 60% of a particular type of insect. What sample size should be used if he desires to be 95% confident that he is within 0.02 of the true fraction of insects killed?
Answer:

$$n = \frac{z^2_{\frac{\alpha}{2}}\hat{p}(1-\hat{p})}{E^2} = \frac{1.96^2(0.60)(1-0.60)}{0.02^2} = 2304.96 \approx 2305.$$

Example 9
Let $10.5, 11.3, 12.8, 9.6, 5.3$ the times in seconds needed for downloading 5 files on your computer from a course website. If we assume that this sample we selected from a normal distribution, construct a 98% confidence interval for the population men μ. Also, construct a 98% confidence interval for the population variance σ^2.
Answer: We need $\bar{x}, s^2$:

```
x <- c(10.5, 11.3, 12.8, 9.6, 5.3)
> mean(x); var(x); sd(x)
[1] 9.9
[1] 7.995
[1] 2.827543
```

98% Confidence interval for the population mean:

$$\bar{x} \pm t_{\frac{\alpha}{2};n-1}\frac{s}{\sqrt{n}} \Rightarrow 9.9 \pm 3.747\frac{2.83}{\sqrt{5}} \Rightarrow 9.9 \pm 4.74 \Rightarrow 5.16 \leq \mu \leq 14.64.$$

98% Confidence interval for the population variance:

$$\left[\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2};n-1}}, \quad \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2};n-1}}\right] \Rightarrow \left[\frac{(5-1)\times 7.995}{13.28}, \frac{(5-1)\times 7.995}{0.297}\right] \Rightarrow (2.41, 107.7)\,.$$

Example 10
The sample mean lifetime of $n_1 = 100$ light bulbs was found to be equal $\bar{x}_1 = 1500$ hours. After new material was used in the production, another sample of size $n_2 = 100$ light bulbs was selected and gave $\bar{x}_2 = 1600$ hours. If assume that the standard deviation is $\sigma = 150$ hours in both case, construct a 95% confidence interval for the difference in the population means, $\mu_1 - \mu_2$.
Answer:

$$\bar{x}_1 - \bar{x}_2 - z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$1500 - 1600 \pm 1.96\sqrt{\frac{150^2}{100} + \frac{150^2}{100}} \Rightarrow -100 \pm 41.6 \Rightarrow -141.6 \leq \mu_1 - \mu_2 \leq -58.4.$$