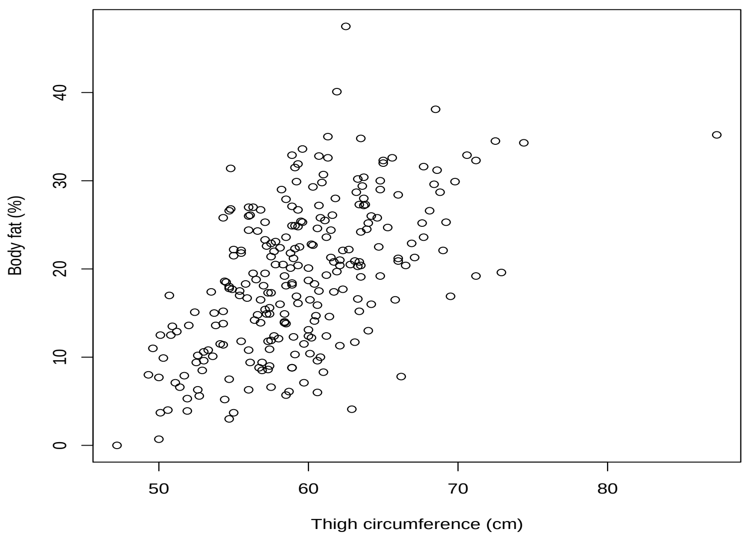
University of California, Los Angeles
Department of Statistics

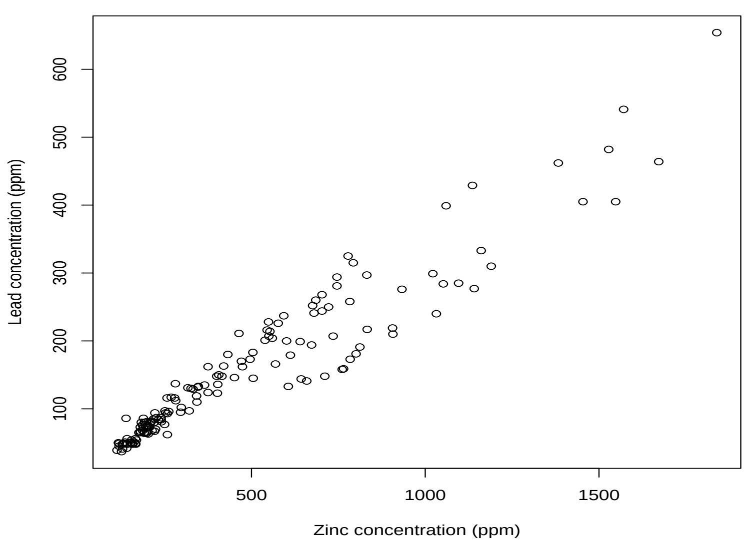
Statistics 100B **Instructor: Nicolas Christou**

Simple regression analysis - summary

Introduction:
Regression analysis is a statistical method aiming at discovering how one variable is related to another variable. It is useful in predicting one variable from another variable. Consider the following "scatterplot" of the percentage of body fat against thigh circumference (cm). This data set is described in detail in the handout on `R`.



And another one: This is the concentration of lead against the concentration of zinc (see handout on `R` for more details on this data set).



What do you observe?

Is there an equation that can model the picture above?

- Regression model equation:

  $$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

  where

  - y response variable (random)
  - x predictor variable (non-random)
  - β_0 intercept (non-random)
  - β_1 slope (non-random)
  - ϵ random error term, $\epsilon \sim N(0, \sigma)$

- Using the method of least squares we estimate β_0 and β_1:

  $$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\sum_{i=1}^n x_i y_i - \frac{1}{n}\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)}{\sum_{i=1}^n x_i^2 - \frac{\left(\sum_{i=1}^n x_i\right)^2}{n}}$$

  $$\hat{\beta}_0 = \frac{\sum_{i=1}^n y_i}{n} - \hat{\beta}_1 \frac{\sum_{i=1}^n x_i}{n} \Rightarrow \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

- The fitted line is:

  $$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

- Distribution of $\hat{\beta}_1$ and $\hat{\beta}_0$:

  $$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}}\right), \quad \hat{\beta}_0 \sim N\left(\beta_0, \sigma\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}}\right)$$

- The standard deviation σ is unknown and it is estimated with the "standard error of the estimate" or "residual standard error" which measures the variability around the fitted line. It is computed as follows:

  $$s_e = \sqrt{\frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n-2}} = \sqrt{\frac{\sum_{i=1}^n e_i^2}{n-2}} = \sqrt{\frac{\sum_{i=1}^n e_i^2}{n-2}}$$

  where

  $e_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i$ is called the residual (the difference between the observed y_i value and the fitted value $\hat{y}_i$.

- Coefficient of determination:
  The total variation in y (total sum of squares $SST = \sum_{i=1}^{n}(y_i - \bar{y})^2$) is equal to the regression sum of squares ($SSR = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$) plus the error sum of squares ($SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$):

$$SST = SSR + SSE$$

  The percentage of the variation in y that can be explained by x is called coefficient of determination (R^2):

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST} \quad \text{Always } \ 0 \leq R^2 \leq 1$$

- Useful:

$$SST = \sum_{i=1}^{n}(y_i - \bar{y})^2 \Rightarrow SST = (n-1)s_y^2 \quad \text{where } s_y^2 \text{ is the variance of } y.$$

- Coefficient of correlation (r):

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

  Or easier for calculations:

$$r = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sqrt{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}\sqrt{\sum_{i=1}^{n} y_i^2 - \frac{\left(\sum_{i=1}^{n} y_i\right)^2}{n}}}$$

  Always $-1 \leq r \leq 1$ and $R^2 = r^2$.

- Another formula for r:

$$r = \hat{\beta}_1 \frac{s_x}{s_y}$$

  where s_x, s_y are the standard deviations of x and y.

- Covariance between y and x:

$$\text{cov}(x, y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

  Therefore

$$r = \frac{\text{cov}(x, y)}{s_x s_y} \Rightarrow \text{cov(x,y)} = r s_x s_y \ \text{ and } \ \hat{\beta}_1 = r \frac{s_y}{s_x}$$

- Standard error of $\hat{\beta}_1$ and $\hat{\beta}_0$:

$$s_{\hat{\beta}_1} = \frac{s_e}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}} = \frac{s_e}{\sqrt{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}}$$

and

$$s_{\hat{\beta}_0} = s_e\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}} = s_e\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}}$$

- Testing for linear relationship between y and x:

$$\begin{aligned} H_0 &: \quad \beta_1 = 0 \\ H_a &: \quad \beta_1 \neq 0 \end{aligned}$$

Test statistic:

$$t = \frac{\hat{\beta}_1 - \beta_1}{s_{\hat{\beta}_1}}$$

Reject H_0 (i.e. there is linear relationship) if $t > t_{\frac{\alpha}{2};n-2}$ or $t < -t_{\frac{\alpha}{2};n-2}$

- Useful things to know:

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n} \quad \text{and} \quad \sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 - \frac{\left(\sum_{i=1}^{n} y_i\right)^2}{n}$$

Simple regression analysis - A simple example

The data below give the mileage per gallon (Y) obtained by a test automobile when using gasoline of varying octane (x):

y	x	xy	y^2	x^2
13.0	89	1157.0	169.00	7921
13.5	93	1255.5	182.25	8649
13.0	87	1131.0	169.00	7569
13.2	90	1188.0	174.24	8100
13.3	89	1183.7	176.89	7921
13.8	95	1311.0	190.44	9025
14.3	100	1430.0	204.49	10000
14.0	98	1372.0	196.00	9604
$\sum_{i=1}^{8} y_i = 108.1$	$\sum_{i=1}^{8} x_i = 741$	$\sum_{i=1}^{8} x_i y_i = 10028.2$	$\sum_{i=1}^{8} y_i^2 = 1462.31$	$\sum_{i=1}^{8} x_i^2 = 68789$

a. **Find the least squares estimates of $\hat{\beta}_0$ and $\hat{\beta}_1$.**

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}} = \frac{10028.2 - \frac{1}{8}(741)(108.1)}{68789 - \frac{741^2}{8}} = 0.100325.$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = \frac{108.1}{8} - 0.100325\frac{741}{8} = 4.2199.$$

Therefore the fitted line is: $\hat{y}_i = 4.2199 + 0.100325 x_i$.

b. **Compute the fitted values and residuals.**

Using the fitted line $\hat{y}_i = 4.2199 + 0.100325 x_i$ we can find the fitted values and residuals. Foe example, the first fitted value is: $\hat{y}_1 = 4.2199 + 0.100325(89) = 13.1488$, and the first residual is $e_1 = y_1 - \hat{y}_1 = 13.0 - 13.1488 = -0.14888$, etc. The table below shows all the fitted values and residuals.

$\hat{y}_i$	e_i	e_i^2
13.14883	-0.14882	0.02215
13.55013	-0.05013	0.00251
12.94818	0.05183	0.00269
13.24915	-0.04915	0.00242
13.14883	0.15118	0.02285
13.75078	0.04922	0.00242
14.25240	0.04760	0.00227
14.05175	-0.05175	0.00268
	$\sum_{i=1}^{n} = 0$	$\sum_{i=1}^{n} e_i^2 = 0.05998$

c. **Find the estimate of σ^2.**

$$s_e^2 = \frac{\sum_{i=1}^{n} e_i^2}{n-2} = \frac{0.05998}{8-2} = 0.009997.$$

Therefore, $s_e = \sqrt{0.009997} = 0.09999$.

d. **Compute the standard error of $\hat{\beta}_1$.**

$$s_{\hat{\beta}_1} = \frac{s_e}{\sqrt{\sum_{i=1}^n x_i^2 - \frac{\left(\sum_{i=1}^n x_i\right)^2}{n}}} = \frac{0.09999}{\sqrt{68789 - \frac{741^2}{8}}} = 0.00806.$$

e. **Estimate the miles per gallon for an octane gasoline level of 94.**

$$\hat{y} = 4.2199 + 0.100325(94) = 13.65.$$

f. **Compute the coefficient of determination, R^2.**

$$R^2 = 1 - \frac{SSE}{SST} = 1 - \frac{\sum_{i=1}^n e_i^2}{(n-1)s_y^2} = 1 - \frac{0.05998}{7(0.2298)} = 0.9627.$$

Therefore, 96.27% of the variation in Y can be explained by x.

The same example can be done with few simple commands in `R`:

```
#Enter the data:
> x <- c(89,93,87,90,89,95,100,98)
> y <- c(13,13.5,13,13.2,13.3,13.8,14.3,14)

#Run the regression of y on x:
> ex <- lm(y ~x)

#Display the results:
> summary(ex)

Call:
lm(formula = y ~ x)

Residuals:
       Min         1Q     Median         3Q        Max
-0.1488221 -0.0505280 -0.0007717  0.0498781  0.1511779

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  4.21990    0.74743   5.646  0.00132 **
x            0.10032    0.00806  12.447 1.64e-05 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.09999 on 6 degrees of freedom
Multiple R-squared: 0.9627, Adjusted R-squared: 0.9565
F-statistic: 154.9 on 1 and 6 DF,  p-value: 1.643e-05
```