**University of California, Los Angeles**
**Department of Statistics**

**Statistics 13** **Instructor: Nicolas Christou**

**Exam 2 - practice questions solutions**

**Problem 1**

Answer the following questions: Answer the following questions:

a. A pollster is contacting an exit poll at a polling place. Suppose the proportion of voters favoring a certain candidate for these elections is 60%. What is the probability that the 6th supporter of this candidate will be the 13th to be interviewed?
ANSWER:

$$\binom{12}{5} 0.60^5 \times 0.40^7 \times 0.60 = 0.0605.$$

b. An electronic fuse is produced by five production lines in a manufacturing operation. The fuses are costly, are quite reliable, and are shipped to suppliers in 100-unit lots. Because testing is destructive, most buyers of the fuses test only a small number of fuses before deciding to accept or reject lots of incoming fuses. All five production lines usually produce only 2% defective fuses. Unfortunately, production line 1 suffered mechanical difficulty and produced 5% defectives during the previous month. This situation became known to the manufacturer after the fuses had been shipped. A customer received a lot produced last month and tested three fuses. What is the probability that exactly one fuse failed? Use binomial probabilities!
ANSWER:

$$\begin{aligned}
P(X=1) &= P(X=1 \cap L_1) + P(X=1 \cap L_2) + P(X=1 \cap L_3) \\
&+ P(X=1 \cap L_4) + P(X=1 \cap L_5) \\
&= P(X=1|L_1)P(L_1) + P(X=1|L_2)P(L_2) + P(X=1|L_3)P(L_3) \\
&+ P(X=1|L_4)P(L_4) + P(X=1|L_5)P(L_5) \\
&= \binom{3}{1}0.05^1 0.95^2 \times \frac{1}{5} + \binom{3}{1}0.02^1 0.98^2 \times \frac{1}{5} + \binom{3}{1}0.02^1 0.98^2 \times \frac{1}{5} \\
&+ \binom{3}{1}0.02^1 0.98^2 \times \frac{1}{5} + \binom{3}{1}0.02^1 0.98^2 \times \frac{1}{5} = 0.0732.
\end{aligned}$$

c. Refer to question (b). A customer received a lot produced last month and tested three fuses. One fuse failed. What is the probability that the lot produced on line 1? Use binomial probabilities!
ANSWER:

$$P(L_1|X=1) = \frac{P(X=1 \cap L_1)}{P(X=1)} = \frac{\binom{3}{1}0.05^1 0.95^2 \times \frac{1}{5}}{0.0732} = 0.3699.$$

d. The manager of an industrial plant is planning to buy a new machine of type $A$. The number of daily repairs $X_A$ required to maintain a machine of type $A$ is a random variable with mean and variance both equal to $0.10t$, where $t$ denotes the number of hours of daily operation. The daily cost of operating $A$ is $C_A(t) = 10t + 30X_A^2$. Assume that the repairs take negligible time and that each night the machines are tuned so that they operate essentially like new machines at the start of the next day. Find the expected daily cost if a workday consists of 10 hours.
ANSWR:

$$\begin{aligned}
E(C_A) &= E(10t + 30X_A^2) = E(100 + 30X_A^2) = 100 + 30E(X_A^2) = 100 + 30(var(X_A) + (E(X_A))^2) \\
&= 100 + 30(0.10 \times 10 + (0.10 \times 10)^2) = 160.
\end{aligned}$$

**Problem 2**

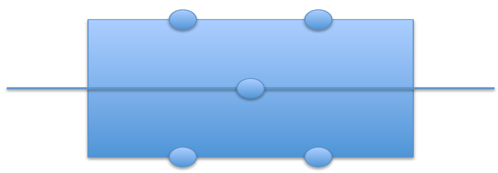
Answer the following questions:

a. You bet \$1 on a single number at a roulette table (it pays 35 to 1). A roulette wheel has 38 slots, numbered 0, 00, and 1 through 36. Use the normal approximation to binomial to approximate the probability that in 1000 bets you win more than 28 times.
ANSWER:
$X \sim b(1000, \frac{1}{38})$, $EX = np = 1000 \times \frac{1}{38} = 26.32$, $\sigma = \sqrt{np(1-p)} = \sqrt{1000 \times \frac{1}{38} \times \frac{37}{38}} = 5.06$.

$$P(X > 28) \approx P(Z > \frac{28.5 - 26.32}{5.06}) = P(Z > 0.43) = 1 - 0.6664 = 0.3336.$$

b. What is the probability that the following system works if each component fails independently with probability $p$? Your answer will be a function of $p$.
ANSWER:
Let $A$ be the event that the top brach works, $B$ be the event that the center branch works, and $C$ the event that the lower branch works. We want $P(A \cup B \cup C)$.

$$\begin{aligned} P(A \cup B \cup C) &= 1 - P(A' \cap B' \cap C') = 1 - P(A')P(B')P(C') \\ &= 1 - \left(1 - (1-p)^2\right)\left(1 - (1-p)^2\right)p = 1 - p^3(2-p)^2. \end{aligned}$$



c. Suppose the prices of objects $A, B$, and $C$ are independent normal random variables with means \$5, \$10, \$15 and standard deviations \$1, \$3, \$5 respectively. Suppose a person wants to buy three of object $A$, four of object $B$, and one of object $C$. Find the probability that the person will spend more than \$100? ANSWER:
The answer is based on the theorem that linear combinations of normal random variables follow the normal distribution. Therefore $3X + 4Y + Z \sim N(70, \sqrt{178})$.

$$P(3X + 4Y + Z > 100) = P(Z > \frac{100 - 70}{\sqrt{178}}) = P(Z > 2.25) = 1 - 0.9878 = 0.0122.$$

d. Suppose that $A$, $B$, and $C$ are three events such that
$A$, $B$ are disjoint,
$A$, $C$ are independent, and
$B$, $C$ are independent.
Suppose also that $4P(A) = 2P(B) = P(C)$, and $P(A \cup B \cup C) = 5P(A)$.
Determine the value of $P(A)$. Also, draw the Venn diagram for these events.
ANSWER:

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= P(A) + 2P(A) + 4P(A) - 4P(A)P(A) - 8P(A)P(A) = 5P(A). \end{aligned}$$

Note: $P(A \cap B) = 0$ and $P(A \cap B \cap C) = 0$ because $A, B$ are disjoint. Also, $P(B) = 2P(A)$ and $P(C) = 4P(A)$.
Finally,

$$7P(A) - 12P(A)^2 = 5P(A) \Rightarrow P(A) = \frac{1}{6}.$$

**Problem 3**

Answer the following questions:

a. A man buys a racehorse for \$20000, and enters it in two races. He plans to sell the horse afterward, hoping to make a profit. If the horse wins both races, its value will jump to \$100000. If it wins one of the races, it will worth \$50000. If it loses both races, it will worth only \$10000. The man believes there is a 20% chance that the horse will win the first race and a 30% chance it will win the second one. Assuming that the two races are independent events, find the man's expected profit.
ANSWER:
We first need to construct the probability distribution of the profit $X$:

| Event | $X$ | $P(X)$ |
|---|---|---|
| Lost both | $-\$10000$ | $0.8 \times 0.7 = 0.56$ |
| Win one | $\$30000$ | $0.2 \times 0.7 + 0.8 \times 0.3 = 0.38$ |
| Win both | $\$80000$ | $0.2 \times 0.3 = 0.06$ |

Therefore the expected profit is:
$E(X) = -10000 \times 0.56 + 30000 \times 0.38 + 80000 \times 0.06 = \$10600$.

b. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(50, 10)$. Find $n$ such that $P(\bar{X} > 45.89) = 0.95$. ANSWER:
The answer is based on the distribution of the sample mean: $\bar{X} \sim N(50, \frac{10}{\sqrt{n}})$ and $Z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$.

$$-1.645 = \frac{45.89 - 50}{\frac{10}{\sqrt{n}}} \Rightarrow n \approx 16.$$

c. Refer to question (b). Find $n$ such that $P(1152 < T < 1348) = 0.95$.
ANSWER:
The answer is based on the distribution of the total: $T \sim N(n\mu, \sigma\sqrt{n})$ and $Z = \frac{T-n\mu}{\sigma\sqrt{n}}$.

$$\begin{aligned} 1.96 &= \frac{1348 - 50n}{10\sqrt{n}} \\ 19.6\sqrt{n} &= 1348 - 50n \\ 19.6^2 n &= 2500n^2 - 134800n + 1348^2 \\ 2500n^2 - 135184.2n + 1348^2 &= 0 \end{aligned}$$

Solve to get $n = 25$. Also acceptable answer (much simpler!) is to find the middle value between (1152, 1348) which is equal to 1250. This will be the mean of $T$. Therefore $1250 = 50n \Rightarrow n = 25$.

d. Suppose that in a certain population individual's heights are normally distributed with mean $\mu = 70$ inches and standard deviation $\sigma = 3$ inches. What is the distribution of the heights in centimeters? In feet? Note: 1 inch=2.54 centimeters, 1 foot=12 inches.
ANSWER:
The answer is based on the theorem that if $X \sim N(\mu, \sigma)$ then $Y = aX \sim N(a\mu, a\sigma)$, where $a$ is a constant.
Distribution in centimeters: $N(2.54 \times 70, 2.54 \times 3)$ or $N(177.8, 7.62)$.
Distribution in feet: $N(\frac{1}{12} \times 70, \frac{1}{12} \times 3)$ or $N(5.83, 0.25)$.

**Problem 4**

Answer the following questions:

a. Suppose in a shipment of 10 transistors 2 are defective. You select without replacement 3 transistors. Find the probability distribution of the number of defective transistors among the 3 you select.
ANSWER:

| $X$ | $P(X)$ |
|---|---|
| 0 | $\frac{\binom{2}{0}\binom{8}{3}}{\binom{10}{3}} = 0.4667.$ |
| 1 | $\frac{\binom{2}{1}\binom{8}{2}}{\binom{10}{3}}$=0.4667. |
| 2 | $\frac{\binom{2}{2}\binom{8}{1}}{\binom{10}{3}}$=0.0667 |

b. Chebyshev's inequality provides a probability lower bound as follows: Let $X$ be a random variable with mean $\mu$ and standard deviation $\sigma$. The inequality states that the probability the random variable $X$ falls within $k$ standard deviations from the mean $\mu$ is at least $1 - \frac{1}{k^2}$, i.e., $P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$. This can be useful when we don't know what distribution $X$ follows. Suppose $k = 2$. Using Chebyshev's inequality compute the probability that the random variable $X$ falls within 2 standard deviations from the mean $\mu$. If the random variable $X$ was $N(\mu, \sigma)$ compute the probability that $X$ falls within 2 standard deviations from the mean $\mu$. Does Chebyshev's inequality give a good approximation in the case when $X$ is normal?
ANSWER:
Using Chebyshev's inequality with $k = 2$ the probability will be at least 75%.
Using normal distribution with $k = 2$ the probability will be approximatly 95%.

Chebyshev's inequality does not give a good approximation.

c. The rainfall $X$ (in inches) at a certain location per year follows $N(13, 3)$. Find the probability that in the next 10 years this location will receive rainfall more than 15 inches on at least one year.
ANSWER:
We first find the probability that in any year the rainfall exceeds 15 inches

$$P(X > 15) = P(Z > \frac{15 - 13}{3}) = P(Z > 0.67) = 1 - 0.7486 = 0.2514.$$

The probability that in the next year this location will receive rainfall more than 13 inches in at least one year is $1 - 0.7486^{10} = 0.9447$.

d. When can the standard deviation of $X \sim b(n, p)$ be equal to its mean? Find a relationship between $n$ and $p$ that will make this happen.
ANSWER:
This can happen if $\sqrt{np(1-p)} = np$. Solve to get $n = \frac{1-p}{p}$.
Example: $p = 0.1, n = \frac{0.9}{0.1} = 9$. It follows that $\mu = \sigma = 0.9$.