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Statistics 403 **Instructor: Nicolas Christou**

Neyman Pearson lemma

So far we discussed specific tests for a number of situations $(\mu, p, \sigma^2, \mu_1 - \mu_2, p_1 - p_2)$, etc. But how did we decide on which test statistic to use and how do we know that we have chosen the best rejection region? The answer is given by the Neyman Pearson Lemma described below.

Suppose we wish to test the simple hypothesis
$H_0 : \theta = \theta_0$
against the alternative simple hypothesis
$H_a : \theta = \theta_a$.
As always, a sample of $X_1, X_2, \ldots, X_n$ is selected from a probability distribution with unknown parameter θ.

Let $L(\theta_0)$ denote the likelihood function when $\theta = \theta_0$ and $L(\theta_a)$ denote the likelihood function when $\theta = \theta_a$. Then for a given significance level α, the test that maximizes the power has a rejection region determined by

$$\frac{L(\theta_0)}{L(\theta_a)} < k, \text{ where } k \text{ is some constant.}$$

This test will be the most powerful test for testing H_0 against H_a.

The previous result applies to simple hypotheses. Usually one of the two hypotheses is composite. For example: $H_0 : \theta = \theta_0$
against the alternative composite hypothesis
$H_a : \theta > \theta_0$.
We say that a test that is most powerful for every simple alternative in H_a is uniformly most powerful.

Example 1:
Let X be a single observation from the probability density function $f(x) = \theta x^{\theta-1}, 0 < x < 1$. Find the most powerful test using significance level $\alpha = 0.05$ for testing
$H_0 : \theta = 1$
$H_a : \theta = 2$.

Example 2:
Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma)$, with known σ^2. Find the uniformly most powerful test using significance level α for testing
$H_0 : \mu = \mu_0$
$H_a : \mu > \mu_0$.

Example 3:
Let $X \sim exp(\frac{1}{\lambda})$. Therefore, $f(x) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x}, \lambda > 0, x > 0$. Let $X_1, X_2, \ldots, X_n$ be a random sample from this distribution.

a. Show that the best critical region for testing
   $H_0 : \lambda = 3$
   $H_a : \lambda = 5$
   is based on $\sum_{i=1}^{n} x_i$.

b. If $n = 12$ and using $\frac{2}{\lambda} \sum_{i=1}^{n} x_i \sim \chi^2_{24}$ find the best critical region when the significance level $\alpha = 0.05$.