Inverses of special matrices

Example A
Consider the equal correlation model, \( \text{cov}(Y_i, Y_j) = \rho \sigma^2 \). The variance covariance matrix is of the form \( (a - b)I + bJ \), where \( a = 1, b = \rho, J = 11' \). Therefore, in our model \( \Sigma = \sigma^2 [(1 - \rho)I + \rho J] \). The inverse of this special matrix can be obtained as follows:
\[
\Sigma^{-1} = \frac{1}{\sigma^2(1 - \rho)} \left[ I - \frac{\rho}{1 + (n - 1)\rho} J \right]
\]
The GLS estimator of \( \mu \) is \( \hat{\mu} = 1' \Sigma^{-1} Y \) with variance \( \text{var}(\hat{\mu}) = \frac{\sigma^2}{n} (1 + (n - 1)\rho) \). (We have seen this using a different method!)

Question:
Show that the variance of the error of prediction is \( \sigma_{pred}^2 = \sigma^2 \left[ 1 + \frac{1}{n} \times c \right] \). Find the constant \( c \) (it will be a function of \( n \) and \( \rho \)) and find the condition on \( \rho \) that will make the prediction variance \( \sigma_{pred}^2 \) in the correlated data more precise than the \( \text{var}(Y_p - \bar{Y}) = \sigma^2 (1 + \frac{1}{n}) \) in the i.i.d. data.

Example B
Consider now the following: \( \text{var}(Y_i) = \sigma^2 \) and \( \text{var}(Y_i, Y_j) = \sigma^2 \rho^{|i-j|} \). The inverse of the variance covariance matrix is given as follows:
\[
\Sigma^{-1} = \frac{1}{\sigma^2(1 - \rho^2)} \begin{pmatrix}
1 & -\rho & 0 & 0 & \ldots & 0 \\
-\rho & 1 + \rho^2 & -\rho & 0 & \ldots & 0 \\
0 & -\rho & 1 + \rho^2 & -\rho & \ldots & 0 \\
0 & 0 & -\rho & 1 + \rho^2 & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \ddots & -\rho \\
0 & 0 & \ldots & 0 & -\rho & 1
\end{pmatrix}
\]

Question:
Show that \( \text{var}(\hat{\mu}) = \sigma^2 \frac{1 + \rho}{n-2(1-\rho)+2} \).
Explain what happens to \( \text{var}(\hat{\mu}) \) when \( \rho = 0 \) and when \( \rho = 1 \).

Question:
For the same sample size \( n \), the variance of \( \hat{\mu} \) in the correlated data is less precise than the variance of \( \hat{\mu} = \bar{Y} \) in the i.i.d. case. Show that \( \text{var}(\hat{\mu}) \) is more variable than \( \text{var}(\bar{Y}) \) by a factor of \( \left[ \frac{1-\rho}{1+\rho} + \frac{2}{n} \frac{\rho}{1+\rho} \right]^{-1} \).

Construct a table that shows this extra variability in \( \text{var}(\hat{\mu}) \) by using \( \rho = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 \), and \( n = 1, 2, 3, \ldots, 50 \).