Exercise 1:
Assume that the random variables $Z_1(s)$ and $Z_2(s)$ are measured at locations $s_i, i = 1, \cdots, n$, with $Z_1(s)$ being the target variable. By using the following definition of the cross-variogram

$$2\gamma_{12}(h) = E(Z_1(s) - Z_2(s + h))^2$$

show that the cokriging system can be formulated in terms of the variogram by minimizing

$$\min_{w_1, w_2} \sum_{i=1}^{n} \sum_{k=1}^{n} w_1_i w_1_k \gamma_{11}(s_i - s_k) + 2 \sum_{i=1}^{n} w_1_i \gamma_{11}(s_0 - s_i) - \sum_{i=1}^{n} \sum_{k=1}^{n} w_2_i w_2_k \gamma_{22}(s_i - s_k) + 2 \sum_{i=1}^{n} w_2_i \gamma_{12}(s_0 - s_i) - 2 \sum_{i=1}^{n} \sum_{k=1}^{n} w_1_i w_2_k \gamma_{12}(s_i - s_k)$$

Hint: Begin by expanding the mean squared error:

$$\min E\left( Z_1(s_0) - \sum_{i=1}^{n} w_1_i Z_1(s_i) - \sum_{i=1}^{n} w_2_i Z_2(s_i) \right)^2$$

Exercise 2:
Access the Maas river data at:

```r
a <- read.table("http://www.stat.ucla.edu/~nchristo/statistics_c173_c273/soil1.txt", header=TRUE)
```

a. Perform cokriging predictions on a grid (use by=50), using log(lead) as the target variable and co-located variables log(cadmium), log(copper), and log(zinc).

b. Use cross-validation to compare ordinary kriging with cokriging of question (a). Note: Cross-validation for ordinary kriging can be done with the `krige.cv` function, while cross-validation for cokriging can be done using the `gstat.cv` function.

Exercise 3:
Suppose that $Z$ is a second-order stationary process with $E[Z(s)] = 0$ and with covariance function:

$$c(h) = \begin{cases} 
8 - \sqrt{h}, & 0 < h \leq 64 \\
0, & h > 64 \\
10, & h = 0 
\end{cases}$$

a. What is the sill of $Z$?

b. What is the nugget effect of $Z$?

c. Draw this covariance function (approximately). Make sure you place some important numbers on the graph.

d. Write the semi variogram function $\gamma(h)$ that corresponds to the covariance function above.

e. Consider three spatial locations $s_1, s_2, s_3$ with distances $d_{12} = 2, d_{13} = 3, d_{23} = 4$. Compute

$$E[Z(s_1) - Z(s_3)] \mid [Z(s_2) - Z(s_3)]$$
Exercise 4:
Consider the data shown on the rectangle of the previous page. Our goal is to estimate the average of the variable $Z(s)$ of the rectangle $ABCD$. The rectangle is descritized by the points $a, b, c, d, e, f$, and the observed data points are $z(s_1), z(s_2), z(s_3), z(s_4), z(s_5)$ (see the coordinates and figure below). Assume that the process $Z$ is second order stationary with variogram function:

$$\gamma(h) = \begin{cases} 
0 & h = 0 \\
3(1 - e^{-\frac{h}{3}}) & h > 0
\end{cases}$$

a. Write the system of the kriging equations in terms of the variogram that will give you the weights for the estimation of the average of the rectangle $ABCD$. Explain what each part of the equations represent.

b. Compute the covariance between the observed data point $s_1$ and point $a$ of the rectangle $ABCD$.

c. Write the expression that computes the covariance between the observed data point $s_1$ and the average of the rectangle $ABCD$.

Coordinates of the observed data points $s_1, s_2, s_3, s_4, s_5$ and the points $a, b, c, d, e, f$:

<table>
<thead>
<tr>
<th>Point</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>1.0</td>
<td>10</td>
</tr>
<tr>
<td>$s_2$</td>
<td>2.0</td>
<td>30</td>
</tr>
<tr>
<td>$s_3$</td>
<td>5.0</td>
<td>40</td>
</tr>
<tr>
<td>$s_4$</td>
<td>6.0</td>
<td>20</td>
</tr>
<tr>
<td>$s_5$</td>
<td>4.0</td>
<td>12</td>
</tr>
<tr>
<td>$a$</td>
<td>3.5</td>
<td>28</td>
</tr>
<tr>
<td>$b$</td>
<td>3.5</td>
<td>30</td>
</tr>
<tr>
<td>$c$</td>
<td>3.5</td>
<td>32</td>
</tr>
<tr>
<td>$d$</td>
<td>4.0</td>
<td>28</td>
</tr>
<tr>
<td>$e$</td>
<td>4.0</td>
<td>30</td>
</tr>
<tr>
<td>$f$</td>
<td>4.0</td>
<td>32</td>
</tr>
</tbody>
</table>
Exercise 5:
Suppose that we want to predict the percentage of copper at a particular mine. It is known that copper is found along with lead and zinc. Therefore the percentages of lead and zinc may improve the predictions of copper. Let \( s_1, s_2, \cdots, s_n \) be the observed data point locations. The available data at location \( s_i \) are \( z_1(s_i), z_2(s_i), z_3(s_i), \) \( i = 1, 2, \cdots, n \) (where the subscripts 1, 2, 3 represent copper, lead and zinc). Assume that \( E(Z_1(s)) = \mu_1, E(Z_2(s)) = \mu_2, E(Z_3(s)) = \mu_3. \)

a. Write the expression of the cokriging predictor for copper at location \( s_0 \) using lead and zinc as the co-located variables.

b. What are the necessary conditions for the predictor of part (a) to be unbiased?

c. The system of the cokriging equations for this problem can be written in matrix form in terms of the covariance and cross covariances as follows:

\[
\begin{pmatrix}
\Sigma_{11} & \Sigma_{12} & \Sigma_{13} & 1 & 0 & 0 \\
\Sigma_{21} & \Sigma_{22} & \Sigma_{23} & 0 & 1 & 0 \\
\Sigma_{31} & \Sigma_{32} & \Sigma_{33} & 0 & 0 & 1 \\
1' & 0' & 0' & 0 & 0 & 0 \\
0' & 1' & 0' & 0 & 0 & 0 \\
0' & 0' & 1' & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
? \\
? \\
? \\
? \\
? \\
? \\
\end{pmatrix}
= 
\begin{pmatrix}
? \\
? \\
? \\
? \\
? \\
? \\
\end{pmatrix}
\]

i. What do the matrices \( \Sigma_{11}, \Sigma_{12}, \Sigma_{13} \) represent?

ii. Write the missing elements of the two vectors above?