Exercise 1
Answer the following questions:

a. Consider ordinary lognormal kriging. Find the unbiased predictor of \( Z(s_0) \). Note: We predict \( Y(s_0) \), where \( Y(s_0) = \ln(Z(s_0)) \).

Answer:
Using the moment generating function of normal distribution and using your class notes from lognormal simple kriging we find that the unbiased predictor of lognormal ordinary kriging is

\[
\hat{Z}(s_0) = e^{\frac{1}{2}\sigma^2_K}Y(s_0).
\]

b. Consider simple lognormal kriging. We discussed in class the unbiased predictor of \( Z(s_0) \). Note: We predict \( Y(s_0) \), where \( Y(s_0) = \ln(Z(s_0)) \). Find \( E[Z(s_0) - \hat{Z}(s_0)]^2 \), where \( \hat{Z}(s_0) \) is the unbiased predictor of \( Z(s_0) \).

Answer:
In class we discussed that the unbiased predictor of \( Z(s_0) \) is \( \hat{Z}(s_0) = e^{\frac{1}{2}\sigma^2_K}Y(s_0) = e^{\frac{1}{2}\sigma^2_K}\tilde{Y}(s_0) \). Therefore, \( E[Z(s_0) - \hat{Z}(s_0)]^2 = var[Z(s_0) - \hat{Z}(s_0)] = var[Z(s_0) - e^{\frac{1}{2}\sigma^2_K}\tilde{Z}(s_0)] \).

\[
var[Z(s_0) - e^{\frac{1}{2}\sigma^2_K}\tilde{Z}(s_0)] = var(Z(s_0)) + e^{\sigma^2_K}var(\tilde{Z}(s_0)) - 2e^{\frac{1}{2}\sigma^2_K}cov(Z(s_0), \tilde{Z}(s_0)).
\]

Recall that, \( Z(s_0) = e^{Y(s_0)} \) and since \( Y(s_0) \sim N(\mu, \sigma) \) we find using the moment generating function of normal distribution (after setting \( t = 1 \) and \( t = 2 \)) expressions for \( E Z(s_0) \) and \( E Z^2(s_0) \) and therefore we can compute \( var[Z(s_0)] = EZ^2(s_0) - (EZ(s_0))^2 \).

Similarly, \( \hat{Z}(s_0) = e^{\tilde{Y}(s_0)} \) and since \( \tilde{Y}(s_0) \sim N(\mu, \sqrt{\Sigma w}) \) we find using the moment generating function of normal distribution (after setting \( t = 1 \) and \( t = 2 \)) expressions for \( E \hat{Z}(s_0) \) and \( E \hat{Z}^2(s_0) \) and therefore we can compute \( var[\hat{Z}(s_0)] = E \hat{Z}^2(s_0) - (E \hat{Z}(s_0))^2 \).

Finally, \( cov(Z(s_0), \hat{Z}(s_0)) = E(Z(s_0)\hat{Z}(s_0)) - E(Z(s_0)) \times E(\hat{Z}(s_0)) \). We need to find \( E(Z(s_0)\hat{Z}(s_0)) \).

\[
E(Z(s_0)\hat{Z}(s_0)) = E e^{Y(s_0)}e^{\tilde{Y}(s_0)} = E e^{Y(s_0) + \tilde{Y}(s_0)}
\]

Since, \( Y(s_0) + \tilde{Y}(s_0) \sim N(2\mu, \sqrt{\sigma^2 + w^T \Sigma w + 2w^T c}) \), we can use again the moment generating function of normal distribution (by setting \( t = 1 \)) to find \( E(Z(s_0)\hat{Z}(s_0)) \).