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Statistics C173/C273 **Instructor: Nicolas Christou**

Homework 7 - Solutions

Exercise 1

Answer the following questions:

a. Consider ordinary lognormal kriging. Find the unbiased predictor of $Z(s_0)$. Note: We predict $Y(s_0)$, where $Y(s_0) = ln(Z(s_0))$.
Answer:
Using the moment generating function of normal distribution and using your class notes from lognormal simple kriging we find that the unbiased predictor of lognormal ordinary kriging is

$$\check{Z}(s_0) = e^{\frac{1}{2}\sigma^2_{OK} - \lambda + \hat{Y}(s_0)}.$$

b. Consider simple lognormal kriging. We discussed in class the unbiased predictor of $Z(s_0)$. Note: We predict $Y(s_0)$, where $Y(s_0) = ln(Z(s_0))$. Find $E[Z(s_0) - \check{Z}(s_0)]^2$, where $\check{Z}(s_0)$ is the unbiased predictor of $Z(s_0)$.
Answer:
In class we discussed that the unbiased predictor of $Z(s_0)$ is $\check{Z}(s_0) = e^{\frac{1}{2}\sigma^2_{SK} + \hat{Y}(s_0)} = e^{\frac{1}{2}\sigma^2_{SK}} e^{\hat{Y}(s_0)} = e^{\frac{1}{2}\sigma^2_{SK}} \hat{Z}(s_0)$. Therefore, $E[Z(s_0) - \check{Z}(s_0)]^2 = var[Z(s_0) - \check{Z}(s_0)] = var[Z(s_0) - e^{\frac{1}{2}\sigma^2_{SK}} \hat{Z}(s_0)]$.

$$\begin{aligned} var[Z(s_0) - e^{\frac{1}{2}\sigma^2_{SK}} \hat{Z}(s_0)] &= var(Z(s_0)) \\ &+ e^{\sigma^2_{SK}} var(\hat{Z}(s_0)) \\ &- 2e^{\frac{1}{2}\sigma^2_{SK}} cov(Z(s_0), \hat{Z}(s_0)). \end{aligned}$$

Recall that, $Z(s_0) = e^{Y(s_0)}$ and since $Y(s_0) \sim N(\mu, \sigma)$ we find using the moment generating function of normal distribution (after setting $t = 1$ and $t = 2$) expressions for $EZ(s_0)$ and $EZ^2(s_0)$ and therefore we can compute $var[Z(s_0)] = EZ^2(s_0) - (EZ(s_0))^2$.

Similarly, $\hat{Z}(s_0) = e^{\hat{Y}(s_0)}$ and since $\hat{Y}(s_0) \sim N(\mu, \sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}})$ we find using the moment generating function of normal distribution (after setting $t = 1$ and $t = 2$) expressions for $E\hat{Z}(s_0)$ and $E\hat{Z}^2(s_0)$ and therefore we can compute $var[\hat{Z}(s_0)] = E\hat{Z}^2(s_0) - (E\hat{Z}(s_0))^2$.

Finally, $cov(Z(s_0), \hat{Z}(s_0)) = E(Z(s_0)\hat{Z}(s_0)) - E(Z(s_0) \times E(\hat{Z}(s_0)$. We need to find $E(Z(s_0)\hat{Z}(s_0))$.

$$E(Z(s_0)\hat{Z}(s_0)) = Ee^{Y(s_0)}e^{\hat{Y}(s_0)} = Ee^{Y(s_0) + \hat{Y}_(s_0)}$$

Since, $Y(s_0) + \hat{Y}_(s_0) \sim N(2\mu, \sqrt{\sigma^2 + \mathbf{w}'\mathbf{\Sigma}\mathbf{w} + 2\mathbf{w}'\mathbf{c}})$, we can use again the moment generating function of normal distribution (by setting $t = 1$) to find $E(Z(s_0)\hat{Z}(s_0))$.

Exercise 2

The ordinary kriging weights using semivariogram values are given by

$$\mathbf{w} = \mathbf{\Gamma}^{-1}\left[\boldsymbol{\gamma} - \frac{\mathbf{1}'\mathbf{\Gamma}^{-1}\boldsymbol{\gamma} - 1}{\mathbf{1}'\mathbf{\Gamma}^{-1}\mathbf{1}}\mathbf{1}\right]. \tag{1}$$

Suppose we want to predict the first observed data point in the data set. Here, because we are predicting one of the observed values in the data set the first row (and column) of the matrix $\mathbf{\Gamma}$ is exactly the same as the vector $\boldsymbol{\gamma}$. The matrix $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1n} \\ \gamma_{21} & 0 & \gamma_{23} & \cdots & \gamma_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \gamma_{n3} & \cdots & 0 \end{pmatrix} \text{ and } \boldsymbol{\gamma} = \begin{pmatrix} 0 \\ \gamma_{12} \\ \gamma_{13} \\ \vdots \\ \gamma_{1n} \end{pmatrix}.$$

Now partition $\mathbf{\Gamma}$ and $\boldsymbol{\gamma}$ as

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & \boldsymbol{\gamma}^{*\prime} \\ \boldsymbol{\gamma}^{*} & \mathbf{\Gamma}^{*} \end{pmatrix} \text{ and } \boldsymbol{\gamma} = \begin{pmatrix} 0 \\ \boldsymbol{\gamma}^{*} \end{pmatrix},$$

and then use the inverse of a partition matrix (handout #52) to show that from equation (1) we get

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ which means that } \hat{Z}(s_1) = Z(s_1).$$

Note: In equation (1) it will be easier if we first compute $\mathbf{\Gamma}^{-1}\boldsymbol{\gamma}$.