**University of California, Los Angeles**
**Department of Statistics**

**Statistics C173/C273** **Instructor: Nicolas Christou**

## Modeling coregionalization

(Goovaerts, P. (1989), Geostatistics for Natural Resources Evaluation
Isaaks, E.H. and Srivastava, R.M. (1989), Applied Geostatistics)

Suppose there are $k$ colocated random variables. Modeling coregionalization requires computing and fitting of $k+\binom{k}{2} = \frac{k(k+1)}{2}$ auto and cross-semivariograms (or covariance functions).

Suppose $Z_i(s_j), i = 1, \ldots, k$ are $k$ intercorrelated random variables and $s_1, s_2, \ldots, s_n$ are the spatial locations that we observe these variables. Consider a linear combination of the type $Y = \sum_{i=1}^{k} \sum_{j=1}^{n} w_{ji} Z_i(s_j)$. It's variance must be nonegative and therefore,

$$var(Y) = \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{a=1}^{n} \sum_{b=1}^{n} w_{ai} w_{bj} c(s_i, s_j).$$

This variance can be expressed in terms of a matrix $\mathbf{C}$ which contains covariances and cross-covariances involving the $k$ random variables. Therefore, the matrix $\mathbf{C}$ must be positive semidefinite, or equivalently, the matrix of semivariograms must be conditionally negative semidefinite. To achieve this, we express each random variable as a function of $p$ independent random variables each one with mean zero and covariance function $c_i(h)$:

$$Z_i(s) = \sum_{l=1}^{p} a_{il} Y_l + \mu_i, \quad i = 1, \ldots, k.$$

Here they are:

$$\begin{aligned}
Z_1(s) &= a_{11}Y_1 + a_{12}Y_2 + \ldots + a_{1p}Y_p + \mu_1 \\
Z_2(s) &= a_{21}Y_1 + a_{22}Y_2 + \ldots + a_{2p}Y_p + \mu_2 \\
\vdots &= \vdots \\
Z_k(s) &= a_{k1}Y_1 + a_{k2}Y_2 + \ldots + a_{kp}Y_p + \mu_k
\end{aligned}$$

with, $E[Y_l(s)] = 0, l = 1, \ldots, p$.
$E[Z_i(s)] = \mu_i, i = 1, \ldots, k$.
$cov[(Y_l(s), Y_{l'}(s+h)] = c_l(h)$ if $l = l'$, and 0 otherwise.

Using the above we can express the covariance between two random variables $Z_i(s)$ and $Z_i(s+h)$ as follows:

$$\begin{aligned}
c_{12}(h) &= cov[Z_1(h), Z_2(s+h)] \\
&= cov\left(a_{11}Y_1 + a_{12}Y_2 + \ldots + a_{1p}Y_p + \mu_1, a_{21}Y_1 + a_{22}Y_2 + \ldots + a_{2p}Y_p + \mu_2,\right) \\
&= a_{11}a_{21}c_1(h) + a_{12}a_{22}c_2(h) + \ldots + a_{1p}a_{2p}c_p(h).
\end{aligned}$$

Similarly the expression of the cross-semivariogram is:

$$\gamma_{12}(h) = a_{11}a_{21}\gamma_1(h) + a_{12}a_{22}\gamma_2(h) + \ldots + a_{1p}a_{2p}\gamma_p(h).$$

Example:
Consider two random variables $Z_1(s)$ and $Z_2(s)$ and suppose we express them as linear combinations of $Y_1, Y_2, Y_3$.

$$\begin{aligned}
Z_1(s) &= a_{11}Y_1(s) + a_{12}Y_2(s) + a_{13}Y_3(s) + \mu_1 \\
Z_2(s) &= a_{21}Y_1(s) + a_{22}Y_2(s) + a_{23}Y_3(s) + \mu_2
\end{aligned}$$

Then the auto and cross semivariograms can be expressed as follows:

$$\begin{aligned}
\gamma_{11}(h) &= a_{11}^2\gamma_1(h) + a_{12}^2\gamma_2(h) + a_{13}^2\gamma_3(h) \\
\gamma_{22}(h) &= a_{21}^2\gamma_1(h) + a_{22}^2\gamma_2(h) + a_{23}^2\gamma_3(h) \\
\gamma_{12}(h) &= a_{11}a_{21}\gamma_1(h) + a_{12}a_{22}\gamma_2(h) + a_{13}a_{23}\gamma_3(h)
\end{aligned}$$

Or using different notation:

$$\begin{aligned}
\gamma_{11}(h) &= u_1\gamma_1(h) + u_2\gamma_2(h) + u_3\gamma_3(h) \\
\gamma_{22}(h) &= v_1\gamma_1(h) + v_2\gamma_2(h) + v_3\gamma_3(h) \\
\gamma_{12}(h) &= w_1\gamma_1(h) + w_2\gamma_2(h) + w_3\gamma_3(h)
\end{aligned}$$

The semivariograms $\gamma_1(h), \gamma_2(h), \gamma_3(h)$ are called the basic models and we express the previous system of equations for each basic model as follows:

Combinations of the first basic model $\gamma_1(h)$

$$\begin{pmatrix} \gamma_{11}(h)^1 & \gamma_{12}(h)^1 \\ \gamma_{21}(h)^1 & \gamma_{22}(h)^1 \end{pmatrix} = \begin{pmatrix} u_1 & w_1 \\ w_1 & v_1 \end{pmatrix} \begin{pmatrix} \gamma_1(h) & 0 \\ 0 & \gamma_1(h) \end{pmatrix}.$$

Combinations of the second basic model $\gamma_2(h)$

$$\begin{pmatrix} \gamma_{11}(h)^2 & \gamma_{12}(h)^2 \\ \gamma_{21}(h)^2 & \gamma_{22}(h)^2 \end{pmatrix} = \begin{pmatrix} u_2 & w_2 \\ w_2 & v_2 \end{pmatrix} \begin{pmatrix} \gamma_2(h) & 0 \\ 0 & \gamma_2(h) \end{pmatrix}.$$

Combinations of the third basic model $\gamma_3(h)$

$$\begin{pmatrix} \gamma_{11}(h)^3 & \gamma_{12}(h)^3 \\ \gamma_{21}(h)^3 & \gamma_{22}(h)^3 \end{pmatrix} = \begin{pmatrix} u_3 & w_3 \\ w_3 & v_3 \end{pmatrix} \begin{pmatrix} \gamma_3(h) & 0 \\ 0 & \gamma_3(h) \end{pmatrix}.$$

The system will be positive definite if all the matrices of the coefficients $u, v, w$ are positive definite, i.e. $u_j, v_j > 0$ and $u_jv_j > w_j^2$ (determinant is larger than zero).