University of California, Los Angeles
Department of Statistics

Statistics C173/C273 **Instructor: Nicolas Christou**

Results on independence using normally distributed random variables

Suppose $Y_1, \ldots, Y_n$ are i.i.d. random variables with $Y_i \sim N(\mu, \sigma)$. We will show independence between $\sum_{i=1}^n (Y_i - \bar{Y})^2$ and $\bar{Y}$ and that $\sum_{i=1}^n (Y_i - \bar{Y})^2$ can be expressed as a sum of $n-1$ independent linear combinations of $Y_1, \ldots, Y_n$ using the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots\cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{1\times 2}} & \frac{-1}{\sqrt{1\times 2}} & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ \frac{1}{\sqrt{2\times 3}} & \frac{1}{\sqrt{2\times 3}} & \frac{-2}{\sqrt{2\times 3}} & 0 & 0 & \cdots\cdots & 0 & 0 \\ \frac{1}{\sqrt{3\times 4}} & \frac{1}{\sqrt{3\times 4}} & \frac{1}{\sqrt{3\times 4}} & \frac{-3}{\sqrt{3\times 4}} & 0 & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{(n-1)\times n}} & \frac{1}{\sqrt{(n-1)\times n}} & \frac{1}{\sqrt{(n-1)\times n}} & \frac{1}{\sqrt{(n-1)\times n}} & \frac{1}{\sqrt{(n-1)\times n}} & \cdots\cdots & \frac{1}{\sqrt{(n-1)\times n}} & \frac{-(n-1)}{\sqrt{(n-1)\times n}} \end{pmatrix}$$

We see that $\mathbf{A}'\mathbf{A} = \mathbf{I}$. Why?
Consider the expression $\mathbf{AY}$. This is equal to:

$$\mathbf{AY} = \begin{pmatrix} \sqrt{n}\bar{Y} \\ Q_1 \\ Q_2 \\ \vdots \\ \vdots \\ Q_{n-1} \end{pmatrix}. \text{ Therefore, } \quad \mathbf{Y}'\mathbf{Y} = \mathbf{Y}'\mathbf{A}'\mathbf{AY} = (\mathbf{AY})'(\mathbf{AY}) = n\bar{Y}^2 + Q_1^2 + Q_2^2 + \ldots + Q_{n-1}^2.$$

Note:

$$\begin{aligned} Q_1 &= \frac{Y_1 - Y_2}{\sqrt{2}}. \\ Q_2 &= \frac{Y_1 + Y_2 - 2Y_3}{\sqrt{2 \times 3}}. \\ Q_3 &= \frac{Y_1 + Y_2 + Y_3 - 3Y_4}{\sqrt{3 \times 4}}. \\ \vdots & \quad \vdots \\ Q_{n-1} &= \frac{Y_1 + Y_2 + Y_3 + \ldots + Y_{n-1} - (n-1)Y_n}{\sqrt{(n-1) \times n}} \end{aligned}$$

Now consider $\sum_{i=1}^n (Y_i - \bar{Y})^2$:

$$\begin{aligned} \sum_{i=1}^n (Y_i - \bar{Y})^2 &= \sum_{i=1}^n Y_i^2 - n\bar{Y}^2 \\ &= n\bar{Y}^2 + Q_1^2 + Q_2^2 + \ldots + Q_{n-1}^2 - n\bar{Y}^2 \\ &= Q_1^2 + Q_2^2 + \ldots + Q_{n-1}^2 = \sum_{i=1}^{n-1} Q_i^2. \end{aligned}$$