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Statistics C173/C273 **Instructor: Nicolas Christou**

Random vectors and properties

We will continue now by finding the expected value and variance of $\hat{\boldsymbol{\beta}}$. Before we do this, we will review very quickly in the next section the mean and variance of random vectors with some properties.

Mean and variance of a random vector

Let $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$ be a random vector with $E\mathbf{Y} = \begin{pmatrix} EY_1 \\ EY_2 \\ \vdots \\ EY_n \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} = \boldsymbol{\mu}$. The variance covariance matrix of $\mathbf{Y}$ denoted with $var(\mathbf{Y})$ is defined as follows:

$$
\begin{aligned}
var(\mathbf{Y}) &= E(\mathbf{Y}-\boldsymbol{\mu})(\mathbf{Y}-\boldsymbol{\mu})' \\
&= E\begin{pmatrix} Y_1-\mu_1 \\ Y_2-\mu_2 \\ \vdots \\ Y_n-\mu_n \end{pmatrix}(Y_1-\mu_1, Y_2-\mu_2, \ldots, Y_n-\mu_n) \\
&= E\begin{pmatrix} (Y_1-\mu_1)^2 & (Y_1-\mu_1)(Y_2-\mu_2) & \ldots & (Y_1-\mu_1)(Y_n-\mu_n) \\ (Y_2-\mu_2)(Y_1-\mu_1) & (Y_2-\mu_2)^2 & \ldots & (Y_2-\mu_2)(Y_n-\mu_n) \\ \vdots & \vdots & \ddots & \vdots \\ (Y_n-\mu_n)(Y_1-\mu_1) & (Y_n-\mu_n)(Y_2-\mu_2) & \ldots & (Y_n-\mu_n)^2 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_n^2 \end{pmatrix} = \boldsymbol{\Sigma}.
\end{aligned}
$$

So $\boldsymbol{\Sigma}$ is the variance covariance matrix of the vector $\mathbf{Y}$. It is symmetric and positive definite. Two important results are given below that will helps us find the expected value and variance of $\hat{\boldsymbol{\beta}}$.

1. Expected value and variance of a linear combination of $\mathbf{Y}$. Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ be a vector of constants and let $q = \mathbf{a}'\mathbf{Y}$. Then $E(q) = E(\mathbf{a}'\mathbf{Y}) = \mathbf{a}'E(\mathbf{Y}) = \mathbf{a}'\boldsymbol{\mu}$. The variance of q can be found as follows:

$$
\begin{aligned}
var(q) = E(q-\mu_q)^2 &= E(\mathbf{a}'\mathbf{Y} - \mathbf{a}'\boldsymbol{\mu})^2 \\
&= E(\mathbf{a}'\mathbf{Y} - \mathbf{a}'\boldsymbol{\mu})(\mathbf{a}'\mathbf{Y} - \mathbf{a}'\boldsymbol{\mu}) \\
&= \mathbf{a}'E(\mathbf{Y}-\boldsymbol{\mu})(\mathbf{Y}-\boldsymbol{\mu})'\mathbf{a} \\
&= \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}.
\end{aligned}
$$

   Note: q is a scalar and therefore its variance should be a scalar and not a matrix. We can verify that $var(q) = \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}$ is 1×1.

2. Let $\mathbf{A}$ be a $p \times n$ matrix of constants. We will examine now $\mathbf{Q} = \mathbf{AY}$. Unlike q (see result (1) above), $\mathbf{Q}$ is a $p \times 1$ vector and therefore its variance should be a $p \times p$ matrix. Let's find the expected value of $\mathbf{Q}$ first. $E(\mathbf{Q}) = E(\mathbf{AY}) = \mathbf{A}E(\mathbf{Y}) = \mathbf{A}\boldsymbol{\mu}$.

$$
\begin{aligned}
var(\mathbf{Q}) = E(\mathbf{Q} - E(\mathbf{Q}))(\mathbf{Q} - E(\mathbf{Q}))' &= E(\mathbf{AY} - \mathbf{A}\boldsymbol{\mu})(\mathbf{AY} - \mathbf{A}\boldsymbol{\mu})' \\
&= \mathbf{A}E(\mathbf{Y}-\boldsymbol{\mu})(\mathbf{Y}-\boldsymbol{\mu})'\mathbf{A}' \\
&= \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'.
\end{aligned}
$$