Random vectors and properties

We will continue now by finding the expected value and variance of \( \hat{\beta} \). Before we do this, we will review very quickly in the next section the mean and variance of random vectors with some properties.

Mean and variance of a random vector

Let \( \mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \) be a random vector with \( E \mathbf{Y} = \begin{pmatrix} EY_1 \\ EY_2 \\ \vdots \\ EY_n \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} = \mathbf{\mu} \). The variance covariance matrix of \( \mathbf{Y} \) denoted with \( \text{var}(\mathbf{Y}) \) is defined as follows:

\[
\text{var}(\mathbf{Y}) = E(\mathbf{Y} - \mathbf{\mu})(\mathbf{Y} - \mathbf{\mu})'
\]

\[
= E \begin{pmatrix} Y_1 - \mu_1 \\ Y_2 - \mu_2 \\ \vdots \\ Y_n - \mu_n \end{pmatrix} \begin{pmatrix} Y_1 - \mu_1 & Y_2 - \mu_2 & \cdots & Y_n - \mu_n \\ Y_1 - \mu_1 & (Y_2 - \mu_2)^2 & \cdots & (Y_n - \mu_n)(Y_2 - \mu_2) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1 - \mu_1 & (Y_2 - \mu_2) & \cdots & (Y_n - \mu_n)^2 \end{pmatrix}
\]

\[
= E \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix} = \mathbf{\Sigma}.
\]

So \( \mathbf{\Sigma} \) is the variance covariance matrix of the vector \( \mathbf{Y} \). It is symmetric and positive definite. Two important results are given below that will help us find the expected value and variance of \( \hat{\beta} \).

1. Expected value and variance of a linear combination of \( \mathbf{Y} \). Let \( \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \) be a vector of constants and let \( q = \mathbf{a}' \mathbf{Y} \). Then \( E(q) = E(\mathbf{a}' \mathbf{Y}) = \mathbf{a}' E(\mathbf{Y}) = \mathbf{a}' \mathbf{\mu} \). The variance of \( q \) can be found as follows:

\[
\text{var}(q) = E(q - \mu_q)^2 = E(\mathbf{a}' \mathbf{Y} - \mathbf{a}' \mathbf{\mu})^2
\]

\[
= E(\mathbf{a}' \mathbf{Y} - \mathbf{a}' \mathbf{\mu})(\mathbf{a}' \mathbf{Y} - \mathbf{a}' \mathbf{\mu})'
\]

\[
= \mathbf{a}' E(\mathbf{Y} - \mathbf{\mu})(\mathbf{Y} - \mathbf{\mu})' \mathbf{a}
\]

\[
= \mathbf{a}' \mathbf{\Sigma} \mathbf{a}.
\]

Note: \( q \) is a scalar and therefore its variance should be a scalar and not a matrix. We can verify that \( \text{var}(q) = \mathbf{a}' \mathbf{\Sigma} \mathbf{a} \) is \( 1 \times 1 \).

2. Let \( \mathbf{A} \) be a \( p \times n \) matrix of constants. We will examine now \( \mathbf{Q} = \mathbf{A} \mathbf{Y} \). Unlike \( q \) (see result (1) above), \( \mathbf{Q} \) is a \( p \times 1 \) vector and therefore its variance should be a \( p \times p \) matrix. Let’s find the expected value of \( \mathbf{Q} \) first. \( E(\mathbf{Q}) = E(\mathbf{A} \mathbf{Y}) = \mathbf{A} E(\mathbf{Y}) = \mathbf{A} \mathbf{\mu} \).

\[
\text{var}(\mathbf{Q}) = E(\mathbf{Q} - E(\mathbf{Q}))(\mathbf{Q} - E(\mathbf{Q})')
\]

\[
= E(\mathbf{A} \mathbf{Y} - \mathbf{A} \mathbf{\mu})(\mathbf{A} \mathbf{Y} - \mathbf{A} \mathbf{\mu})'
\]

\[
= \mathbf{A} \mathbf{\Sigma} \mathbf{A}.'