Problem 1 (20 points)
The betas for 10 stocks in two historical periods 2000-2004 and 2005-2009 are as follows:

\[
\begin{array}{c|cc}
\text{beta1} & \text{beta2} \\
[1,] & 0.9072828 & 0.7333601 \\
[2,] & 1.0874136 & 1.0096048 \\
[3,] & 0.9871119 & 1.1143148 \\
[4,] & 1.0084073 & 1.1011334 \\
[5,] & 0.7606293 & 0.7711888 \\
[6,] & 0.8047901 & 0.7834646 \\
[7,] & 0.9533157 & 0.9914738 \\
[8,] & 0.8036708 & 1.1083840 \\
[9,] & 1.0667607 & 0.9524100 \\
[10,] & 1.0315184 & 0.8759303 \\
\end{array}
\]

a. Explain how you can obtain an estimate for the beta of stock 8 for the period 2010-2014 using the Blume's technique.

b. Suppose that for the second period 2005-2009 the variance of the return of the S&P500 index is \( \sigma^2_w = 0.00217 \). Assume that the single index model holds. Find the covariance between stocks 1 and 3 during the same period.

\[
\sigma_{13} = \beta_1 \beta_3 \sigma^2_w = (0.733)(1.114)(0.00217) = 0.00177
\]

c. Explain how you can obtain an estimate for the beta of stock 8 for the period 2010-2014 using the Vচčěk's technique.

d. Suppose that the correlation coefficient between stock A and S&P500 during the period 2005-2009 is 0.20. The variance of the return of stock A during the same period is 0.0143 and the variance of the return of the S&P500 index was \( \sigma^2_w = 0.00217 \). Find the beta of stock A.
\[ \bar{R}_G = \bar{R}_F + \left( \frac{\bar{R}_G - \bar{R}_F}{\sigma_G} \right) \sigma_p \]

\[ \sigma = \frac{1}{0.08 - 0.02} \]

\[ z_A = \frac{1}{0.9 (0.2)} \left( \frac{0.08}{0.04} - C \right) \]

\[ z_B = \frac{1}{0.9 (0.08)} \left( \frac{\bar{R}_G - 0.04}{0.08} - C \right) \]

\[ \frac{1}{0.9 (0.2)} \left[ \frac{0.08}{0.2} - C \right] = \frac{1}{0.9 (0.08)} \left[ \frac{\bar{R}_G - 0.04}{0.08} - C \right] \]

\[ \frac{0.08}{0.04} - \frac{C}{0.1} = \frac{\bar{R}_G - 0.04}{0.0064} - \frac{C}{0.08} \]
Problem 2 (20 points)
Use the following for questions (a) and (b) below:

\[
\begin{pmatrix}
0.04 & 0.0016 \\
0.0016 & 0.0064
\end{pmatrix}
\begin{pmatrix}
Z_A \\
Z_B
\end{pmatrix}
= \begin{pmatrix}
0.12 - 0.04 \\
0.016 - 0.0064
\end{pmatrix}
\begin{pmatrix}
\bar{R}_A \\
\bar{R}_B
\end{pmatrix}
= \begin{pmatrix}
25.25 - 6.1 \\
-6.1 - 1.828
\end{pmatrix}
\begin{pmatrix}
0.08 \\
0.04
\end{pmatrix}
\]

It is also given that \( \rho_{AB} = 0.1 \).

a. What expected return on stock B would result in an optimum portfolio of \( \frac{1}{2} A \) and \( \frac{1}{2} B \)? Assume short sales are allowed and that \( R_f = 0.04 \).

Since \( X_A = X_B = \frac{1}{2} \Rightarrow Z_A = Z_B \)

\[
\begin{align*}
0.12 - 0.04 &= 0.04 Z_A + 0.0016 Z_B \\
0.016 - 0.0064 &= 0.0016 Z_A + 0.0064 Z_B
\end{align*}
\]

\[
\frac{Z_A}{Z_B} = \frac{0.04}{0.0016} \Rightarrow Z_A = \frac{0.08}{0.04} \Rightarrow Z_A = 2
\]

\[
\bar{R}_B = 0.04 + \left( 0.0016 + 0.0064 \right) \cdot 2 = 0.08 + 0.04 \Rightarrow \bar{R}_B = 0.08 + 0.04 \Rightarrow \bar{R}_B = 0.12
\]

b. What expected return on stock B would mean that stock B would not be held? Assume short sales are allowed and that \( R_f = 0.04 \).

\[
X_B = 0 \Rightarrow Z_B = 0
\]

\[
\begin{align*}
0.12 - 0.04 &= 0.04 Z_A + 0.0016 Z_B \\
0.016 - 0.0064 &= 0.0016 Z_A + 0.0064 Z_B
\end{align*}
\]

\[
\frac{Z_A}{Z_B} = \frac{0.04}{0.0016} \Rightarrow Z_A = \frac{0.08}{0.04} \Rightarrow Z_A = 2
\]

\[
\bar{R}_B = 0.04 + 0.0016 \cdot 2 = 0.04 + 0.0032 \Rightarrow \bar{R}_B = 0.04 + 0.0032 \Rightarrow \bar{R}_B = 0.0432
\]

c. Suppose \( X \) and \( Y \) represent the returns of two stocks. Show that these two random variables \( X \) and \( Y \) cannot possibly have the following properties: \( E(X) = 0.3, E(Y) = 0.2, E(X^2) = 0.1, E(Y^2) = 0.29 \), and \( E(XY) = 0 \). Reminder: \( \sigma_{XY} = E(X - \mu_X)(Y - \mu_Y) = E(XY) - (E(X))(E(Y)) \).

\[
\begin{align*}
\text{Var}(X) &= 0.1 - 0.3^2 = 0.01 \Rightarrow \text{SD}(X) = 0.1 \\
\text{Var}(Y) &= 0.29 - 0.2^2 = 0.25 \Rightarrow \text{SD}(Y) = 0.5 \\
\text{Cov}(X, Y) &= E(XY) - (E(X))(E(Y)) = 0 - (0.3)(0.2) = 0.06 \\
\rho &= \frac{-0.06}{0.1 \cdot 0.5} = -1.2
\end{align*}
\]

\[-1 \leq \rho \leq 1\]
Data for problem 3:

# Create the ticker vector for the two stocks plus the S&P500:
> ticker <- c("ibm", "xom", "GSPC")
> data <- getReturns(ticker, start="2005-01-01", end="2009-12-31")

# Get the summary statistics:
> summary(data$R)

ibm       xom       "GSPC"
Min. :0.205144 Min. :-0.116543 Min. :-0.1694245
1st Qu.:0.013633 1st Qu.:0.0184671 1st Qu.:0.0184670
Median :0.009613 Median :0.003498 Median :0.0099800
Mean :0.009026 Mean :0.008107 Mean :0.0001331
3rd Qu.:0.050789 3rd Qu.:0.045487 3rd Qu.:0.0277094
Max. :0.129405 Max. :0.233054 Max. :0.0939251

# Get the variance covariance matrix:
> cov(data$R)

ibm  xom "GSPC"
ibm 0.0039985797 0.0004087865 0.0017214346
xom 0.0004087865 0.0035546519 0.00009878328
"GSPC"0.0017214346 0.0009878328 0.0021726295

# Run the regression of the returns of IBM on the returns of S&P500
# and obtain alpha, beta, mse:
> reg1 <- lm(data$R[,1] ~ data$R[,3])
> summary(reg1)$coef[1] -> a
[1] 0.0088928891
> summary(reg1)$coef[2] -> b
[1] 0.7923277
> summary(reg1)$sigma^2 -> s^2
[1] 0.002868861

# Run the regression of the returns of EXXON-Mobil on the returns of S&P500
# and obtain alpha, beta, mse:
> reg2 <- lm(data$R[,2] ~ data$R[,3])
> summary(reg2)$coef[1] -> a
[1] 0.0086046374
> summary(reg2)$coef[2] -> b
[1] 0.4546716
> summary(reg2)$sigma^2 -> s^2
[1] 0.003159995
Problem 3  (20 points)
Using the package stockPortfolio we have obtained the returns of IBM, Exxon-Mobil, and the S&P500 index for the period 2005-01-31 to 2009-12-31. The summary statistics, variance-covariance matrix of the returns, and the regressions of the returns of IBM and Exxon-Mobil on the index are shown on the previous page.

a. Using the single index model compute the variance of the returns of IBM.
\[
\sigma_{IB}^2 = \beta_{IB}^2 \sigma_i^2 = 0.2992 \times (0.02147) + 0.002008 \Rightarrow \sigma_{IB}^2 = 0.004
\]

b. What is the beta of a portfolio that consists of 80% IBM and 20% Exxon-Mobil?
\[
\beta_P = 0.8 \times \beta_{IB} + 0.2 \times \beta_{XM} = 0.8 \times 0.492 + 0.2 \times 0.455 \Rightarrow \beta_P = 0.7246
\]

c. Using the historical variance-covariance matrix of the returns and assuming \( R_f = 0.008 \) we get the following:

\[
\begin{bmatrix}
1.1 \\
\end{bmatrix}
\]

\[
ibm \; 0.256622417 \\
xom \; 0.000559498
\]

1. Explain how these \( z \) values were computed. No calculations, but please be very specific!
\[
Z = \frac{1}{2} \frac{\sum \text{VAR-COVAR MATRIX}}{R = \begin{bmatrix} \beta_{IB} & \beta_{XM} \\ \beta_{XM} & \beta_f \end{bmatrix}}
\]

2. Compute the proportion of the investor's wealth that goes into each stock.
\[
x_{IB} = \frac{0.256622}{0.256622 + 0.000559} \Rightarrow \begin{bmatrix} x_{IB} = 0.9978 \\ x_{XM} = 0.0022 \end{bmatrix}
\]

3. Compute the expected return and standard deviation of the point of tangency. This point should be point \( G \) on the graph below.
\[
\bar{R}_G = 0.9978 \times 0.009026 + 0.0022 \times 0.008107 \Rightarrow \bar{R}_G = 0.00902
\]
\[
\sigma^2_G = \begin{bmatrix} 0.9978 & 0.0022 \\ 0.0022 & 0.0081 \end{bmatrix} \begin{bmatrix} 0.009026 \\ 0.008107 \end{bmatrix} \Rightarrow \sigma_G = 0.00781
\]

\[
x_{IB} \times \sigma_{IB}^2 + x_{XM} \times \sigma_{XM}^2 = 0.00781
\]
Problem 4  (20 points)
Using the single index model three stocks $x$, $y$, $z$ were ranked based on the excess return to beta ratio as follows:

<table>
<thead>
<tr>
<th>Stock</th>
<th>$\frac{R_i - R_f}{\beta_i}$</th>
<th>$(R_i - R_f)\beta_i$</th>
<th>$\sum_{j=1}^{i} \frac{(R_j - R_f)\beta_i}{\sigma_{i,j}^2}$</th>
<th>$\frac{\beta_i^2}{\sigma_{i,i}^2}$</th>
<th>$\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{j,j}^2}$</th>
<th>$C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.0395</td>
<td>13.7170</td>
<td>13.7170</td>
<td>342.2115</td>
<td>347.2115</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$x$</td>
<td>0.0080</td>
<td>4.3307</td>
<td>18.0477</td>
<td>538.4952</td>
<td>885.7067</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$z$</td>
<td>0.0067</td>
<td>1.9733</td>
<td>20.0210</td>
<td>294.4627</td>
<td>1180.1694</td>
<td>$C_3$</td>
</tr>
</tbody>
</table>

Assume $R_f = 2\%$ and that the variance of the returns of the market is $\sigma_m^2 = 0.0023$.

a. Find $C_1, C_2, C_3$.

\[
C_i = \frac{0.0395 \times 13.7170}{1 + 0.0023 \times 347.2115} \Rightarrow C_1 = 0.01754
\]

\[
C_2 = \frac{0.0080 \times 4.3307}{1 + 0.0023 \times 538.4952} \Rightarrow C_2 = 0.01367
\]

\[
C_3 = \frac{0.0067 \times 1.9733}{1 + 0.0023 \times 294.4627} \Rightarrow C_3 = 0.01240
\]

b. What is the composition of the optimal portfolio when short sales are not allowed?

\[
C^* = C_1 = 0.01754
\]

only stock 1. \(\Rightarrow 100\% \text{ in stock } y\).

c. Show the optimal portfolio of part (b) on the graph below. On this graph $x, y, z$ are the three stocks.
Problem 5  (20 points)

Part A:
A portfolio manager wants to present to his clients the efficient frontier using 25 stocks. Explain clearly and in detail how you would help this portfolio manager to trace out the efficient frontier when short sales are allowed but no riskless lending and borrowing exists. You must show a graph, the inputs you are using, the vectors and matrices you are multiplying, etc. One should be able to follow step by step your procedure and be able to trace out the efficient frontier.

Part B:
Suppose short sales are allowed and three stocks \( X, Y, Z \) are used to construct the efficient frontier. Let \( A \) and \( B \) be two portfolios on the efficient frontier with: \( \bar{R}_A = 0.006, \sigma_A = 0.1, \bar{R}_B = 0.01, \sigma_B = 0.2 \) and \( \sigma_{AB} = 0.02 \). The composition of portfolio \( A \) is \( 0.53X, -1.80Y, 2.27Z \). The composition of portfolio \( B \) is \( 0.53X, 1.50Y, 2.27Z \). Find the composition of the minimum risk portfolio in terms of the two portfolios and in terms of the three stocks \( X, Y, Z \). On the previous page draw the graph of the expected return against standard deviation and show approximately the portfolio possibilities curve, identify the efficient frontier, and place the two portfolios \( A, B \), and the minimum risk portfolio on the graph.