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AN ANALYTIC DERIVATION OF THE EFFICIENT PORTFOLIO FRONTIER

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I. Introduction

The characteristics of the mean-variance, efficient portfolio frontier have been discussed at length in the literature.[1] However, for more than three assets, the general approach has been to display qualitative results in terms of graphs.[2] In this paper, the efficient portfolio frontiers are derived explicitly, and the characteristics claimed for these frontiers are verified. The most important implication derived from these characteristics, the separation theorem, is stated and proved in the context of a mutual fund theorem. It is shown that under certain conditions, the classic graphical technique for deriving the efficient portfolio frontier is incorrect.

II. The Efficient Portfolio Set When All Securities Are Risky

Suppose there are m risky securities with the expected return on the i^{th} security denoted by E_i, the covariance of returns between the i^{th} and j^{th} security denoted by σ_{ij}, and the variance of the return on the i^{th} security denoted by $\sigma_{ii} = \sigma_i^2$. If all m securities are assumed risky, $\sigma_i^2 > 0$, $i = 1, \ldots, m$, and if we further assume that no security can be represented as a linear combination of the other securities, then the variance-covariance matrix of returns, $\Omega = [\sigma_{ij}]$, is nonsingular. The frontier of all feasible portfolios which can be constructed from these m securities is defined as the locus of feasible portfolios that have the smallest variance for a prescribed expected return. Let x_i = percentage of the value of a portfolio invested in the i^{th} security,

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[1]See the classical works of Markowitz [10] and Tobin [17 and 18]. See Sharpe [16] for a modern treatment and additional references. The limited validity of the mean-variance assumption is discussed in Borch [2], Feldstein [5], Hakansson [6], and Samuelson [14]. Samuelson [15] has shown that mean-variance is a good approximation for "compact" distributions. Further, Merton [11 and 12] has shown that mean-variance type analysis is valid in intertemporal portfolio problems when trading takes place continuously and asset price changes are continuous.

[2]Exceptions to this graphical analysis are the discussions of equilibrium models: see, for example, Fama [4], Black [1], Mossin [13], and Lintner [8 and 9].

$i=1, \ldots, m$, and as a definitional result, $\Sigma_1^m x_i = 1$. Then, the frontier can be described as the set of portfolios that satisfy the constrained minimization problem,

$$\min \frac{1}{2} \sigma^2 \tag{1}$$

subject to

$$\sigma^2 = \Sigma_1^m \Sigma_1^m x_i x_j \sigma_{ij}$$

$$E = \Sigma_1^m x_i E_i$$

$$1 = \Sigma_1^m x_i,$$

where σ^2 is the variance of the portfolio on the frontier with expected return equal to E.[3] Using Lagrange multipliers, (1) can be rewritten as,

$$\min \{\frac{1}{2} \Sigma_1^m \Sigma_1^m x_i x_j \sigma_{ij} + \gamma_1[E - \Sigma_1^m x_i E_i] + \gamma_2[1 - \Sigma_1^m x_i]\} \tag{2}$$

where γ_1 and γ_2 are the multipliers. The standard first-order conditions for a critical point are,

$$0 = \Sigma_1^m x_j \sigma_{ij} - \gamma_1 E_i - \gamma_2, \; i = 1, \ldots, m, \tag{3a}$$

$$0 = E - \Sigma_1^m x_i E_i, \tag{3b}$$

$$0 = 1 - \Sigma_1^m x_i. \tag{3c}$$

Further, the x's that satisfy (3) minimize σ^2 and are unique by the assumption on Ω. System (3) is linear in the x's and hence, we have from (3a) that

$$x_k = \gamma_1 \Sigma_1^m v_{kj} E_j + \gamma_2 \Sigma^m{}_1 v_{kj}, \; k = 1, \ldots, m, \tag{4}$$

where the v_{ij} are defined as the elements of the inverse of the variance-

[3]It is assumed that borrowing and short-selling of all securities is allowed. Hence, the only constraint on the x_i is that they sum to unity.

covariance matrix, i.e., $\Omega^{-1} \equiv [v_{ij}]$. Multiplying (4) by E_k and summing over $k = 1, \ldots, m$, we have that

$$\Sigma_1^m x_k E_k = \gamma_1 \Sigma_1^m \Sigma_1^m v_{kj} E_j E_k + \gamma_2 \Sigma_1^m \Sigma_1^m v_{kj} E_k, \tag{5}$$

and summing (4) over $k = 1, \ldots, m$, we have that

$$\Sigma_1^m x_k = \gamma_1 \Sigma_1^m \Sigma_1^m v_{kj} E_j + \gamma_2 \Sigma_1^m \Sigma_1^m v_{kj}. \tag{6}$$

Define:

$A \equiv \Sigma_1^m \Sigma_1^m v_{kj} E_j$; $\quad B \equiv \Sigma_1^m \Sigma_1^m v_{kj} E_j E_k$;

$C \equiv \Sigma_1^m \Sigma_1^m v_{kj}$.

From (3b), (3c), (5), and (6), we have a simple linear system for γ_1 and γ_2,

$$\begin{aligned} E &= B\gamma_1 + A\gamma_2 \\ 1 &= A\gamma_1 + C\gamma_2 \end{aligned} \tag{7}$$

where we note that $\Sigma_1^m \Sigma_1^m v_{kj} E_j = \Sigma_1^m \Sigma_1^m v_{kj} E_k$ and that $B > 0$ and $C > 0$.[4] Solving (7) for γ_1 and γ_2, we find that

$$\begin{aligned} \gamma_1 &= \frac{(CE - A)}{D} \\ \gamma_2 &= \frac{(B - AE)}{D} \end{aligned} \tag{8}$$

where $D \equiv BC - A^2 > 0$.[5] We can now substitute for γ_1 and γ_2 from (8) into (4)

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[4] Ω is a nonsingular variance-covariance matrix and, therefore, is symmetric and positive definite. It follows directly that Ω^{-1} is also. Hence, $v_{kj} = v_{jk}$ for all j and k, and B and C are quadratic forms of Ω^{-1} which means that they are strictly positive (unless all $E_i = 0$).

[5] Because Ω^{-1} is positive definite, $0 < \Sigma_1^m \Sigma_1^m v_{ij}(AE_i - B)(AE_j - B) = B^2C - 2A^2B + A^2B = B(BC - A^2) = BD$. But $B > 0$, hence $D > 0$.

to solve for the proportions of each risky asset held in the frontier portfolio with expected return E; namely,

$$x_k = \frac{E\ \Sigma_1^m v_{kj}\ (CE_j - A) + \Sigma_1^m v_{kj}\ (B - AE_j)}{D},\ k = 1, \ldots, m. \tag{9}$$

Multiply (3a) by x_i and sum from i = 1, ..., m to derive

$$\Sigma_i^m \Sigma_i^m x_i x_j\ \sigma_{ij} = \gamma_1\ \Sigma_1^m x_i E_i + \gamma_2\ \Sigma_1^m x_i. \tag{10}$$

From the definition of σ^2, (3b), and (3c), (10) implies

$$\sigma^2 = \gamma_1 E + \gamma_2. \tag{11}$$

Substituting for γ_1 and γ_2 from (8) into (11), we write the equation for the variance of a frontier portfolio as a function of its expected return, as

$$\sigma^2 = \frac{CE^2 - 2AE + B}{D}. \tag{12}$$

Thus, the frontier in mean-variance space is a parabola. Examination of the first and second derivatives of (12) with respect to E shows that σ^2 is a strictly convex function of E with a unique minimum point where $\frac{d\sigma^2}{dE} = 0$, i.e.,

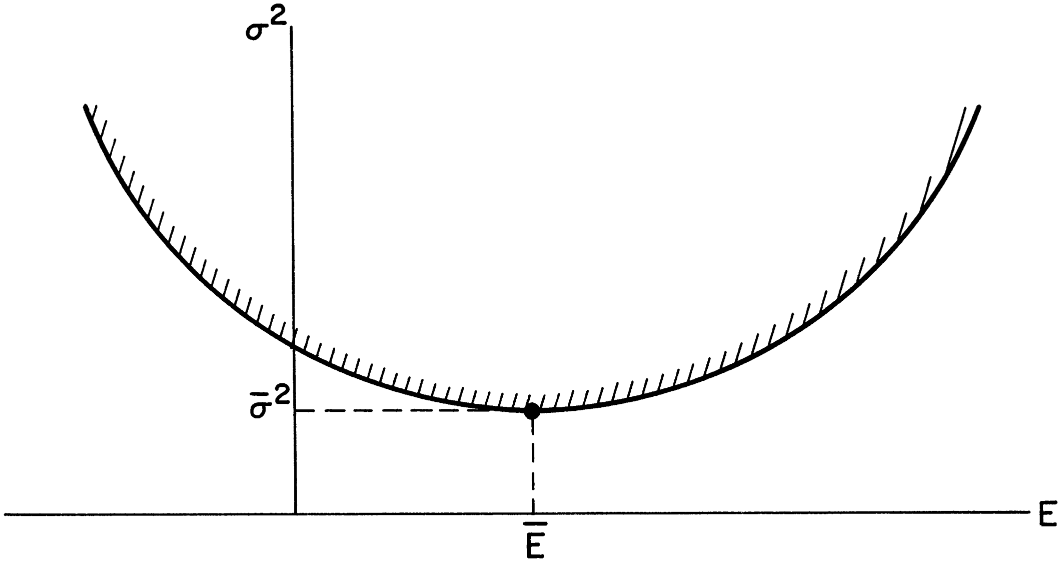
$$\begin{aligned} \frac{d\sigma^2}{dE} &= \frac{2[CE - A]}{D} \\ &= 0 \text{ when } E = \frac{A}{C} \\ \frac{d^2\sigma^2}{dE^2} &= \frac{2C}{D} > 0. \end{aligned} \tag{13}$$

Figure I is a graph of (12) where $\bar{E} \equiv A/C$ and $\bar{\sigma}^2 \equiv 1/C$ are the expected return and variance of the minimum-variance portfolio. Define $\bar{x}_k$ to be the proportion of the minimum-variance portfolio invested in the k^{th} asset, then from (9),

$$\bar{x}_k = \frac{\Sigma_1^m v_{kj}}{C},\ k = 1, \ldots, m. \tag{14}$$

FIGURE I

MEAN-VARIANCE PORTFOLIO FRONTIER:
RISKY ASSETS ONLY



It is usual to present the frontier in the mean-standard deviation plane instead of the mean-variance plane. From (12) and (13), we have that

$$\sigma = \sqrt{(CE^2 - 2AE + B)/D} \tag{15}$$

$$\frac{d\sigma}{dE} = \frac{(CE - A)}{D\sigma}$$

$$\frac{d^2\sigma}{dE^2} = \frac{1}{D\sigma^3} > 0.$$

From (15), σ is a strictly convex function of E, and the minimum standard deviation portfolio is the same as the minimum-variance portfolio. Figure II graphs the frontier which is a hyperbola, in the standard form with E on the ordinate and σ on the abscissa. The broken lines are the asymptotes of the frontier whose equations are

$$E = \bar{E} \pm \sqrt{\frac{D}{C}}\,\sigma. \tag{16}$$

The efficient portfolio frontier (the set of feasible portfolios that have the largest expected return for a given standard deviation) is the heavy-lined part of the frontier in Figure II, starting with the minimum-variance portfolio and moving to the northeast. The equation for E as a function of σ along the frontier is

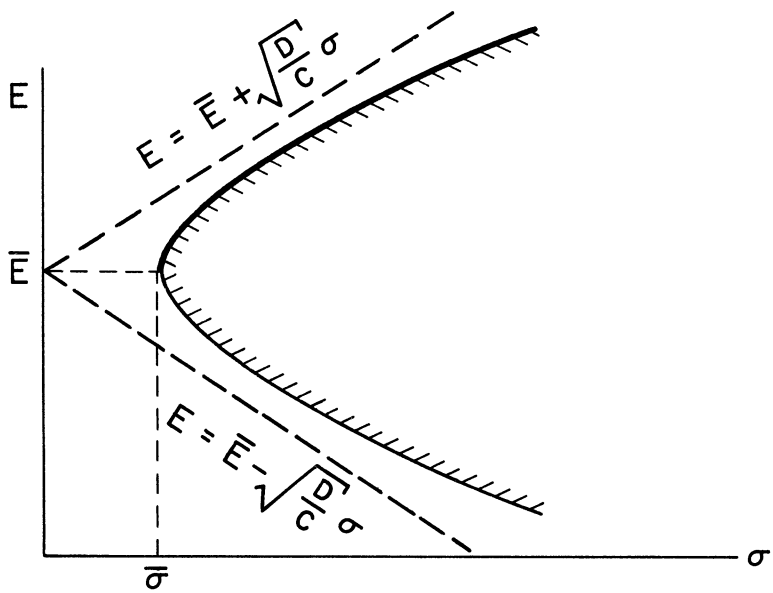
$$\begin{aligned} E &= \frac{A}{C} \pm \frac{1}{C}\sqrt{D(C\sigma^2 - 1)} \\ &= \bar{E} \pm \frac{1}{C}\sqrt{D(C\sigma^2 - 1)}\;. \end{aligned} \tag{17}$$

The equation for the efficient portfolio frontier is

$$E = \bar{E} + \frac{1}{C}\sqrt{DC(\sigma^2 - \bar{\sigma}^2)} \tag{18}$$

FIGURE II

MEAN-STANDARD DEVIATION PORTFOLIO FRONTIER:
RISKY ASSETS ONLY



III. A Mutual Fund Theorem[6]

Theorem I. Given m assets satisfying the conditions of Section II, there are two portfoltios ("mutual funds") constructed from these m assets, such that all risk-averse individuals, who choose their portfolios so as to maximize utility functions dependent only on the mean and variance of their portfolios, will be indifferent in choosing between portfolios from among the original m assets or from these two funds.

To prove Theorem I, it is sufficient to show that any portfolio on the efficient frontier can be attained by a linear combination of two specific portfolios because an optimal portfolio for any individual (as described in the theorem) will be an efficient portfolio.

Equation (9) describes the proportion of the frontier portfolio, with expected return E, invested in the k^{th} asset, $k = 1, \ldots, m$. If we define

$$(19) \qquad g_k \equiv \Sigma_1^m v_{kj} \, (CE_j - A)/D, \qquad k = 1, \ldots, m$$

$$h_k \equiv \Sigma_1^m v_{kj} \, (B - AE_j)/D, \qquad k = 1, \ldots, m,$$

then (9) can be rewritten compactly as

$$(20) \qquad x_k = Eg_k + h_k, \qquad k = 1, \ldots, m.$$

Note that, by their definitions, $\Sigma_1^m g_k = 0$ and $\Sigma_1^m h_k = 1$.

Because we want all individuals to be able to construct their optimal portfolios from just two funds, the proportions of risky assets held by each fund must be independent of preferences (or equivalently, independent of E) and the proportions of the two funds chosen by the investor must be independent of the individual securities' expected returns, variances, and covariances. Let a_k be the proportion of the first fund's value invested in the k^{th} asset, and let b_k be the proportion of the second fund's value invested in the k^{th} asset

[6]For a general discussion of mutual fund or "separation" theorems, see Cass and Stiglitz [3]. A theorem similar to Theorem I is proved in Merton [11] for intertemporal portfolio decisions when asset returns are lognormally distributed, and investors have concave utility functions. One advantage of the mutual fund description of the separation theorem over the classical graphical interpretation comes when the analysis is extended to an intertemporal context, where generally more than two portfolios ("funds") are required to span the space of investors' optimal portfolios (i.e., generalized separation). In this case, it is quite natural to interpret each portfolio as a mutual fund providing a "service" to the investor while graphical description is impossible. See, for example, Merton [12].

($\Sigma_1^m a_k = \Sigma_1^m b_k = 1$), and a_k and b_k must satisfy

$$x_k = Eg_k + h_k = \lambda a_k + (1 - \lambda)b_k, \qquad k = 1, \ldots, m, \tag{21}$$

where λ is the particular "mix" of the funds that generates the efficient portfolio with expected return E.[7] All solutions to (21) will have $\lambda = \delta E - \alpha$ where δ and α are constants ($\delta \neq 0$) that depend on the expected returns of the two funds, E_a and E_b, respectively. Substituting for λ in (21) and imposing the condition that a_k and b_k be independent of E, we have that a_k and b_k must satisfy

$$\begin{aligned} g_k &= \delta(a_k - b_k) \\ h_k &= b_k - \alpha(a_k - b_k), \qquad k = 1, \ldots, m. \end{aligned} \tag{22}$$

For $\delta \neq 0$, (22) can be solved for a_k and b_k to give

$$\begin{aligned} a_k &= b_k + g_k/\delta \\ b_k &= h_k + \alpha g_k/\delta, \qquad k = 1, \ldots, m. \end{aligned} \tag{23}$$

Factors a and b are two linearly independent vectors that form a basis for the vector space of frontier portfolios, x.[8] Two portfolios whose holdings satisfy (23) will be called a set of basis portfolios. Two such portfolios must be frontier portfolios although they need not be efficient. Hence, from (20), both funds holdings are completely determined by their expected returns. Because $E_a \equiv \Sigma_1^m a_k E_k$, $E_b \equiv \Sigma_1^m b_k E_k$, $\Sigma_1^m h_k E_k = 0$, and $\Sigma_1^m g_k E_k = 1$, we have that

$$\begin{aligned} E_a &= \frac{(1 + \alpha)}{\delta} \\ E_b &= \frac{\alpha}{\delta} \end{aligned} \tag{24}$$

Alternatively, given values for E_a and E_b, the constants δ and α can be written

[7] Two funds with proportions a_k and b_k which satisfy (21) will generate all frontier portfolios, including as a subset, the efficient portfolios.

[8] a and b are m-vectors with elements a_k and b_k, $k = 1, \ldots, m$. x is an m-vector with elements x_k, where the x_k satisfy (20).

as

(25) $$\delta = \frac{1}{(E_a - E_b)}$$

$$\alpha = \frac{E_b}{(E_a - E_b)} \quad .$$

Different values for δ and α correspond to nonsingular transformations of one basis set of portfolios into another basis, and as can be seen in (25) their values are independent of preferences. Thus, the investor need only know the means, variances, and covariances of the two funds to determine the mix, λ, that generates his optimal portfolio. The funds' "managers" can choose δ and α arbitrarily ($\delta \neq 0$) and then, follow the investment program prescribed in (23) without knowledge of individual preferences or wealth distribution. Hence Theorem I is proved.

The essential characteristics of a set of basis portfolios are the expected returns, variances, and covariances. Equation (24) describes how the expected returns depend on δ and α. Because both portfolios are frontier portfolios, (24) and (12) can be combined to determine the variance of the first fund, σ_a^2, and of the second fund, σ_b^2:

(26) $$\sigma_b^2 = (C\alpha^2 - 2A\alpha\delta + B\delta^2)/D\delta^2, \text{ and}$$

(27) $$\sigma_a^2 = \sigma_b^2 + [C + 2(\alpha C - A\delta)]/D\delta^2.$$

To find the covariance, σ_{ab}, we use (23) as follows:

(28) $$\begin{aligned}\sigma_{ab} &= \Sigma_1^m \Sigma_1^m a_i b_j \sigma_{ij} \\ &= \Sigma_1^m \Sigma_1^m b_i b_j \sigma_{ij} + \frac{1}{\delta} \Sigma_1^m \Sigma_1^m g_i h_j \sigma_{ij} + \frac{\alpha}{\delta^2} \Sigma_1^m \Sigma_1^m g_i g_j \sigma_{ij} \\ &= \sigma_b^2 - [\frac{A\delta}{C} - \alpha](\frac{C}{D\delta^2}).\end{aligned}$$

Using (26) and (28), we can find those combinations of δ and α that will make the two portfolios uncorrelated (i.e., $\sigma_{ab} = 0$). For $\delta \neq 0$; $\sigma_{ab} = 0$ when

(29) $$C\alpha^2 + B\delta^2 - 2A\alpha\delta + C\alpha - A\delta = 0.$$

(29) is an equation for a conic section, and because $A^2 - BC = -D < 0$, it must be an equation for an ellipse (see Figure III).

If we restrict both portfolios to be efficient[9] and take the convention that $\sigma_a^2 > \sigma_b^2$, then $E_a > E_b \geq \bar{E} = A/C$, and from (25), δ must be positive and $\alpha \geq A\delta/C$. One could show that the line $\alpha = A\delta/C$ is tangent to the ellipse at the point ($\alpha = 0$, $\delta = 0$) as drawn in Figure III. Therefore, two efficient portfolios that are uncorrelated do not exist. From (38), we have that $\sigma_{ab} \geq \sigma_b^2 > 0$, which implies that all efficient portfolios are positively correlated. Further, $\sigma_{ab}^2 = \sigma_b^2$ if and only if $\alpha = A\delta/C$. If $\alpha = A\delta/C$, then, from (24), $E_b = A/C = \bar{E}$ which implies that the portfolio held by the fund with proportions b is the minimum-variance portfolio with $\sigma_b^2 = 1/C$ and $b_k = \bar{x}_k$. Because 1/C is the smallest variance of any feasible portfolio, it must be that, for efficient portfolios, σ_{ab} will be the smallest when one of the portfolios is the minimum-variance portfolio. In this case, the portfolio of the other fund will have the characteristics that

(30)
$$E_a = \frac{1}{\delta} + \frac{A}{C}$$

$$\sigma_a^2 = \frac{1}{C} + \frac{C}{D\delta^2}$$

$$a_k = \frac{\Sigma_1^m v_{kj}}{C} + \frac{g_k}{\delta}, \quad k = 1, \ldots, m,$$

where δ is arbitrary.

There does not appear to be a "natural" choice for the value of δ. However, it will be useful to know the characteristics of the frontier portfolio which satisfies

(31)
$$\frac{dE}{d\sigma} = \frac{E - R}{\sigma}$$

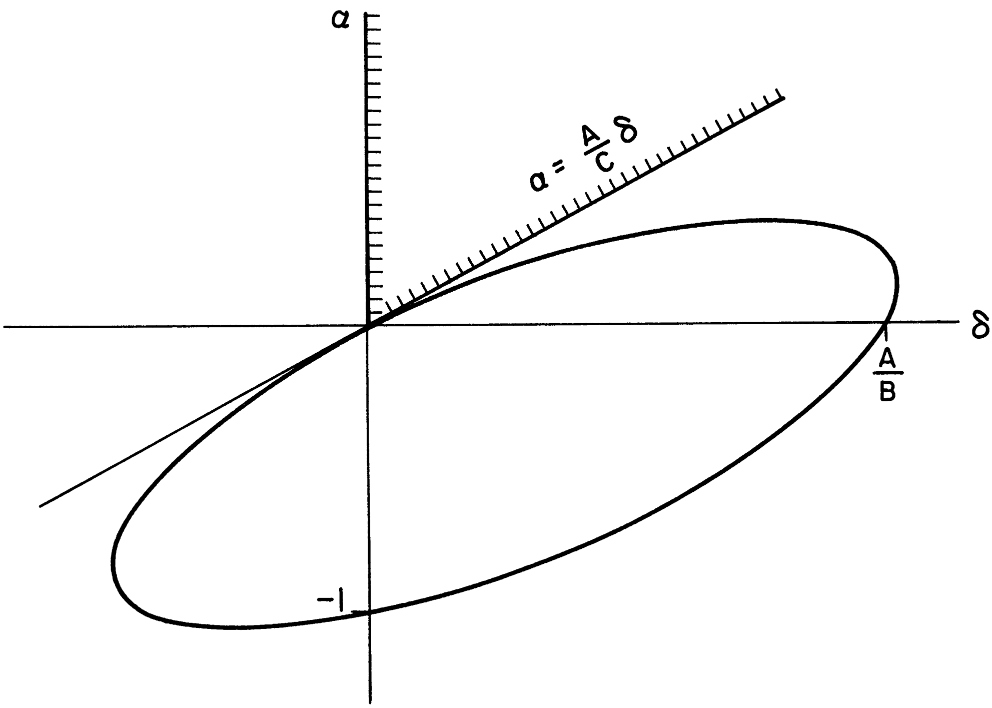
for some given value of R. From (15), $\frac{dE}{d\sigma}$ along the frontier equals $D\sigma/(CE - A)$. If we choose δ such that the portfolio with proportions a satisfies (31), then

(32a)
$$\delta = \frac{C(A - CR)}{D}$$

[9]Although the paper does not impose equilibrium market clearing conditions, it is misleading to allow as one of the mutual funds a portfolio that no investor would hold as his optimal portfolio.

FIGURE III

ZERO-CORRELATION ELLIPSE AMONG FRONTIER PORTFOLIOS



$$a_k = \frac{\Sigma_1^m v_{kj}(E_j - R)}{(A - RC)}, \; k = 1, \ldots, m. \tag{32b}$$

If $R < \overline{E} = A/C$, then $\delta > 0$ and the portfolio is efficient. If $R > \overline{E}$, then $\delta < 0$, and the portfolio is inefficient. If $R = \overline{E}$, $\delta = 0$ and equation (31) cannot be satisfied by *any* frontier portfolio with finite values of E and σ. The implications of these results will be discussed in the following section.

IV. The Efficient Portfolio Set When One of the Assets Is Riskless

The previous sections analyzed the case in which all the available assets are risky. In this section, we extend the analysis to include a riskless asset, by keeping the same m risky assets as before and adding a $(m + 1)^{st}$ asset with a guaranteed return R. In an analogous way to (2) in Section II, the frontier of all feasible portfolios is determined by solving the problem:

$$\min \{\tfrac{1}{2} \Sigma_1^m \Sigma_1^m x_i x_j \sigma_{ij} + \lambda[E - R - \Sigma_1^m x_i(E_i - R)]\}. \tag{33}$$

Notice that the constraint $\Sigma_1^{m+1} x_i = 1$ does not appear in (33) because we have explicitly substituted for $x_{m+1} = 1 - \Sigma_1^m x_i$; i.e., the $x_1, \ldots, x_m$ are unconstrained by virtue of the fact that x_{m+1} can be always chosen such that $\Sigma_1^{m+1} x_i = 1$ is satisfied. This substitution not only simplifies the analytics of solving (33) but also provides insight into some results derived later in the paper.

The first-order conditions derived from (33) are

$$0 = \Sigma_1^m x_j \sigma_{ij} - \lambda(E_i - R), \qquad i = 1, \ldots, m, \tag{34a}$$

$$0 = E - R - \Sigma_1^m x_i(E_i - R). \tag{34b}$$

In a fashion similar to the previous section, we derive the equation for the frontier,

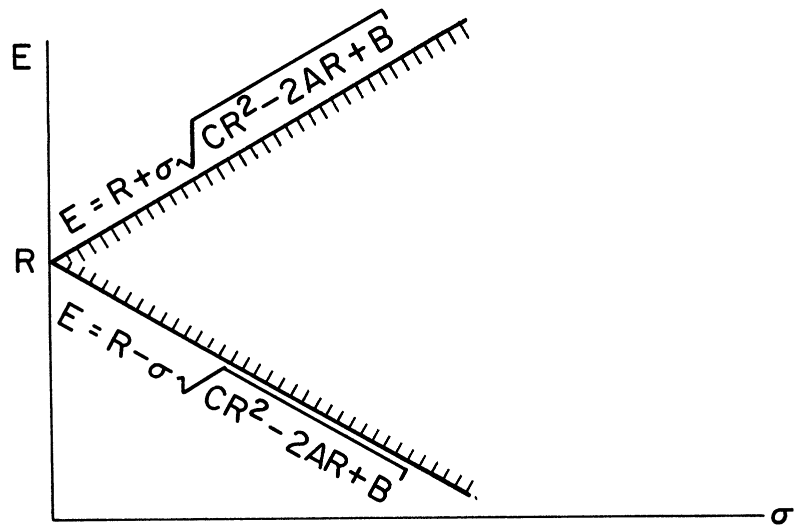
$$|E - R| = \sigma\sqrt{CR^2 - 2AR + B}, \tag{35}$$

which is drawn in Figure IV, and the proportions of risky assets for the frontier portfolios as a function of E are

$$x_k = \frac{(E - R)\, \Sigma_1^m v_{kj}(E_j - R)}{CR^2 - 2AR + B}, \qquad k = 1, \ldots, m. \tag{36}$$

FIGURE IV

MEAN-STANDARD DEVIATION PORTFOLIO FRONTIER



As pictured in Figure IV, the frontier is convex (although not strictly convex), and the efficient locus is that portion of the frontier where $E \geq R$. Since the efficient locus is linear in σ, all efficient portfolios are perfectly correlated. From (35) and (36), the lower (inefficient) part of the frontier represents short sales of the risky holdings of the efficient portfolio with the same σ.

Because all efficient portfolios are perfectly correlated, it is straightforward to show that Theorem I holds in the case in which one of the securities is riskless, by simply selecting any two distinct portfolios on the frontier. However, one usually wants a theorem stronger than Theorem I when one of the assets is riskless; namely, the two mutual funds can be chosen such that one fund holds only the riskless security and the other fund contains only risky assets (i.e., in the notation of the previous section, $a_{m+1} = 0$ and $b_k = 0$, $k = 1, \ldots, m$).

Theorem II. Given m assets satisfying the conditions of Section II and a riskless asset with return R, there exists a unique pair of efficient[10] mutual funds, one containing only risky assets and the other only the riskless asset, such that all risk-averse individuals, who choose their portfolios so as to maximize utility functions dependent only on the mean and variance of their portfolios, will be indifferent in choosing between portfolios from among the original m+1 assets or from these two funds, if and only if $R < \overline{E}$.
The proof of Theorem II follows the approach to proving Theorem I. If

$$u_k \equiv \Sigma_1^m v_{kj} \, (E_j - R)/(CR^2 - 2AR + B), \text{ then}$$

$$\Sigma_1^m u_k = (A - RC)/(CR^2 - 2AR + B).$$

Define $\lambda = \delta(E - R) + 1 - \alpha$, $(\delta \neq 0)$, then we have that

$$x_k = (E - R)\, u_k = \lambda a_k + (1 - \lambda)\, b_k \tag{37}$$

$$= \delta(E - R)(a_k - b_k) + (1 - \alpha)(a_k - b_k) + b_k$$

$$k = 1, \ldots, m,$$

[10]The reasons for considering only efficient portfolios here are stronger than those given in footnote 9. Given that one of the funds holds only the riskless asset, the aggregate demand for a mutual fund with portfolio proportions along the inefficient part of the frontier would have to be negative, which violates the spirit, if not the mathematics, of the mutual fund theorem.

and

$$(38) \qquad x_{m+1} = 1 - (E - R)(A - RC)/(CR^2 - 2AR + B)$$
$$= \delta(E - R)(a_{m+1} - b_{m+1}) + (1 - \alpha)(a_{m+1} - b_{m+1}) + b_{m+1}$$

where $a_{m+1} = 1 - \Sigma_1^m a_k$ and $b_{m+1} = 1 - \Sigma_1^m b_k$. Solving (37) for a_k and b_k, we have that

$$(39) \qquad a_k = \alpha u_k/\delta$$

$$b_k = (\alpha - 1)\, u_k/\delta, \qquad k = 1, \ldots, m$$

and

$$(40) \qquad a_{m+1} = b_{m+1} - (A - RC)/\delta(CR^2 - 2AR + B)$$

$$b_{m+1} = 1 - (\alpha - 1)(A - RC)/\delta(CR^2 - 2AR + B).$$

Now require that one of the funds (say the one with proportions b) hold only the riskless asset (i.e., $b_k = 0$, $k = 1, \ldots, m$ and $b_{m+1} = 1$) which is accomplished by choosing $\alpha = 1$. If it is also required that the other fund hold only risky assets (i.e., $a_{m+1} = 0$), then from (43), $\delta = (A - RC)/(CR^2 - 2AR + I$ Note that if $R = A/C$, $\delta = 0$, which is not allowed, and as can be seen in (40), in this case, $a_{m+1} = b_{m+1} = 1$. From (39), the two mutual funds are different since $b_k = 0$ for all $k = 1, \ldots, m$ and $a_k \neq 0$ for some k. However, $\Sigma_1^m a_k = 0$, which means that the "risky" fund holds a hedged portfolio of long and short positions whose net value is zero. If $R > A/C$, then $\delta < 0$, and the portfolio is inefficient (i.e., $E_a < R$). If $R < A/C$, then $\delta > 0$, and the portfolio is efficient. When $R < A/C$, the composition of the efficient risky portfolio is

$$(41) \qquad a_k = \frac{\Sigma_1^m v_{kj}(E_j - R)}{(A - RC)}, \qquad k = 1, \ldots, m.$$

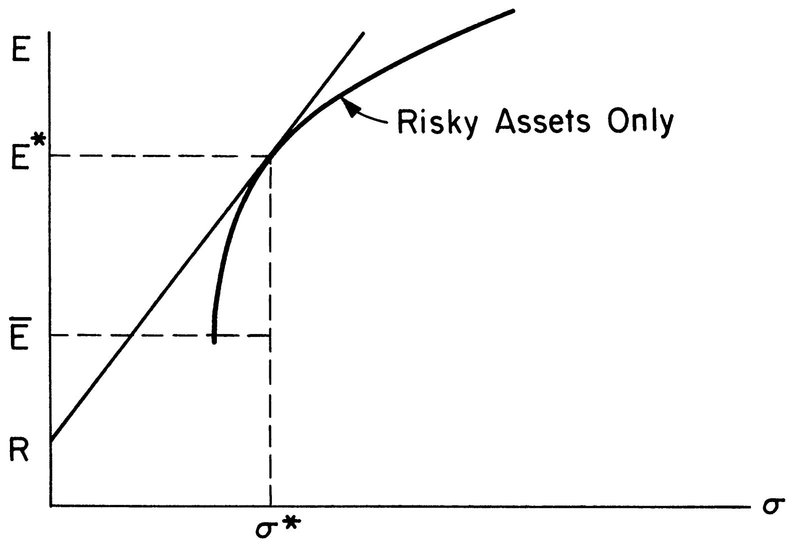
Thus, Theorem II is proved.

The traditional approach to finding the efficient frontier when one of the assets is riskless is to graph the efficient frontier for risky assets only, and then to draw a line from the intercept tangent to the efficient frontier as illustrated in Figure V. Suppose that the point (E^*, σ^*) as drawn in Figure V exists. Then one could choose one mutual fund to be the riskless asset and the

FIGURE V

RELATIONSHIP BETWEEN EFFICIENT FRONTIERS:

$\bar{E} > R$



other to be (E^*, σ^*) which contains only risky assets by virtue of the fact that (E^*, σ^*) is on the efficient frontier for risky assets only. But, by Theorem II, two such mutual funds exist if and only if $R < \bar{E} = A/C$ (as is the case in Figure V). Analytically, the portfolio with expected return and standard deviation, E^* and σ^*, was derived in equations (31) and (32), and the proportions are identical to those in (41)(as they should be).

The proper graphical solutions when $R \geq \bar{E}$ are displayed in Figures VI and VII. When $R = \bar{E}$, there is no tangency for finite E and σ, and the frontier lines (with the riskless asset included) are the asymptotes to the hyperbolic frontier curve for risky assets only. When $R > \bar{E}$, there is a lower tangency and the efficient frontier lies above the upper asymptote. Under no condition can one construct the entire frontier (with the riskless security included) by drawing tangent lines to the upper *and* lower parts of the frontier for risky assets only.[11] The intuitive explanation for this result is that with the introduction of a riskless asset, it is possible to select a portfolio with net nonpositive amounts of risky assets; this was not possible when one could only choose among risky assets.

Although for individual portfolio selection, there is no reason to rule out $R \geq \bar{E}$, one can easily show that as a general equilibrium solution with homogeneous expectations, Figure V is the only possible case with (E^*, σ^*), the market portfolio's expected return and standard deviation. Hence, we have as a necessary condition for equilibrium that $R < \bar{E}$.

Given that the proportions in the market portfolio must be the same as in (41) (i.e., $x_k^M = a_k$, k = 1, ..., m where "M" denotes "for the market portfolio"), the fundamental result of the capital asset pricing model, the security market line, can be derived directly as follows:

$$\text{(42)} \qquad \sigma_{kM} \equiv \Sigma_1^m x_i^M \sigma_{ij}, \; k = 1, \ldots, m$$

$$\equiv \Sigma_1^m (\Sigma_1^m v_{ij}(E_j - R))\, \sigma_{ik}/(A - RC), \text{ from (44)}$$

$$= \Sigma_1^m (E_j - R) \Sigma_1^m v_{ij}\sigma_{ik}/(A - RC)$$

$$= (E_k - R)/(A - RC), \text{ and}$$

[11] There seems to be a tendency in the literature to draw graphs with $R \geq \bar{E}$ and an upper tangency (e.g., Fama [4, p. 26] and Jensen [7, p. 174] In Sharpe [16, Chapter 4], the figures appear to have $R = \bar{E}$ and a *double* tangency.

FIGURE VI

RELATIONSHIP BETWEEN EFFICIENT FRONTIERS:

$\bar{E} = R$

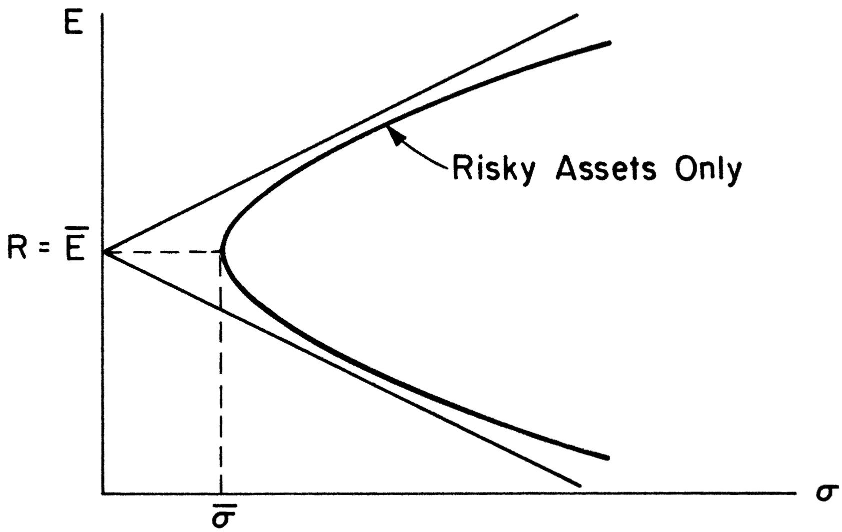
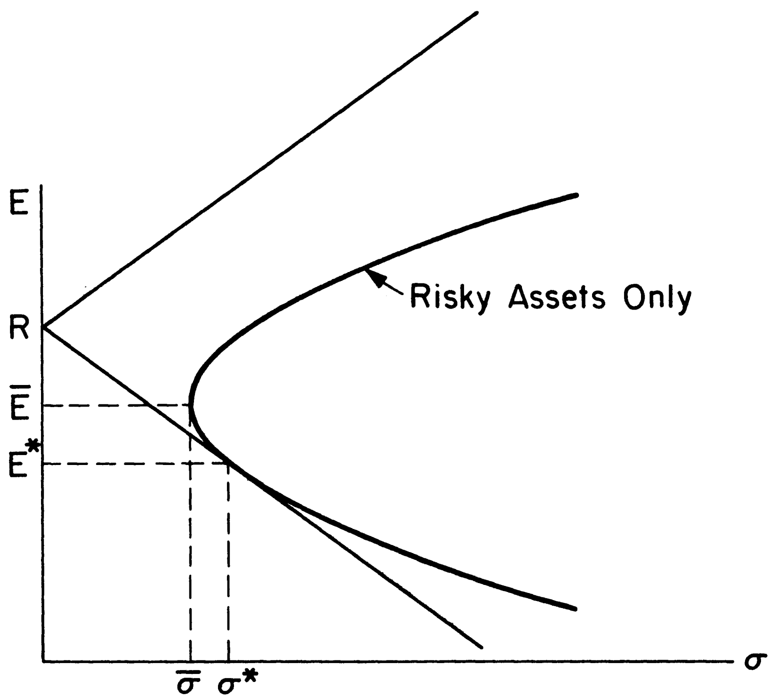


FIGURE VII

RELATIONSHIP BETWEEN EFFICIENT FRONTIERS:

$\bar{E} < R$



(43) $$\sigma_M^2 \equiv \Sigma_1^m x_i^M \sigma_{iM}$$

$$= \Sigma_1^m x_i^M (E_i - R)/(A - RC), \text{ from (42)}$$

$$= (E_M - R)/(A - RC) \ .$$

Eliminating (A - RC) by combining (42) and (43), we derive

(44) $$E_k - R = \frac{\sigma_{kM}}{\sigma 2_M} (E_M - R), \qquad k = 1, \ldots, m,$$

which is the security market line.

REFERENCES

[1] Black, F. "Capital Market Equilibrium with Restricted Borrowing." Forthcoming in *Journal of Business*.

[2] Borch, K. "A Note on Uncertainty and Indifference Curves." *Review of Economic Studies*, 36, January 1969.

[3] Cass, D., and J. Stiglitz. "The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds." *Journal of Economic Theory*, 2, June 1970.

[4] Fama, E. "Risk, Return, and Equilibrium." *Journal of Political Economy*, 79, January-February 1971.

[5] Feldstein, M. S. "Mean-Variance Analysis in the Theory of Liquidity Preference and Portfolio Selection." *Review of Economic Studies*, 36, January 1969.

[6] Hakansson, N. H. "Capital Growth and the Mean-Variance Approach to Portfolio Selection." *Journal of Financial and Quantitative Analysis*, January 197

[7] Jensen, M. "Risk, the Pricing of Capital Assets, and the Evaluation of Investment Portfolios." *Journal of Business*, Vol. 42, April 1969.

[8] Lintner, J. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *Review of Economics and Statistics*, XLVIII, February 1965.

[9] __________. "Security Prices, Risk, and Maximal Gains from Diversification." *Journal of Finance*, March 1968.

[10] Markowitz, H. *Portfolio Selection: Efficient Diversification of Investment*. New York: John Wiley and Sons, 1959.

[11] Merton, R.C. "Optimum Consumption and Portfolio Rules in a Continuous-Time Model." *Journal of Economic Theory*, Vol. 3, December 1971.

[12] __________. "An Intertemporal Capital Asset Pricing Model." Forthcoming in *Econometrica*.

[13] Mossin, J. "Equilibrium in a Capital Asset Market." *Econometrica*, October 1966.

[14] Samuelson, P.A. "General Proof that Diversification Pays." *Journal of Financial and Quantitative Analysis*, Vol. II, March 1967.

[15] __________. "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances, and Higher Moments." *Review of Economic Studies*, 37, October 1970.

[16] Sharpe, W. *Portfolio Theory and Capital Markets*. New York: McGraw-Hill Book Company, 1970.

[17] Tobin, J. "Liquidity Preference as Behavior Toward Risk." *Review of Economic Studies*, Vol. 25, February 1958.

[18] __________. "The Theory of Portfolio Selection." *The Theory of Interest Rates*. Edited by F.H. Hahn and F.P.R. Brechling. New York: Macmillan, 1965.