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Statistics C183/C283 **Instructor: Nicolas Christou**

Distribution of R^2

Consider the multiple regression model with k predictors. (In simple regression $k = 1$). It can be shown that the F test for the overall significance of the model is equal to $\frac{R^2}{1-R^2}\frac{n-k-1}{k}$.

Using the result above show that $R^2 \sim \text{Beta}(\frac{k}{2}, \frac{n-k-1}{2})$.
Hint 1: For easier notation let $R^2 = W$, so $F = \frac{W}{1-W}\frac{n-k-1}{k}$. Solve for W, and use the method of CDF to find the distribution of W.
Hint 2: Let $X \sim F_{n_1,n_2}$. The pdf of the F distribution is

$$f(x) = \frac{\Gamma(\frac{n_1+n_2}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}\left(1+\frac{n_1}{n_2}x\right)^{-\frac{1}{2}(n_1+n_2)}, \quad 0 < x < \infty.$$

Hint 3: As a reminder the beta distribution has the following pdf. Let $X \sim Beta(\alpha, \beta)$ then

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad \alpha > 0,\ \beta > 0,\ 0 \le x \le 1, \text{ where } B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \text{ (beta function).}$$

Proof:

$$F = \frac{R^2}{1-R^2}\frac{n-k-1}{k}$$

$$F = \frac{W}{1-W}\frac{n-k-1}{k} \quad \text{OR} \quad W = \frac{kF}{kF+n-k-1}$$

FIND CDF ON W:

$$F_W(w) = P(W \le w) = P\left(\frac{kF}{kF+n-k-1} \le w\right)$$

$$= P\left(F < \frac{n-k-1}{k(1-w)}w\right)$$

NEED TO FIND PDF OF W. TAKE DERIVATIVE ON BOTH SIDES W.R.T. w

$$f(w) = \frac{(n-k-1)k(1-w)+k(n-k-1)w}{k^2(1-w)^2} \times \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{k}{2})\cdot\Gamma(\frac{n-k-1}{2})} \times \left(\frac{k}{n-k-1}\right)^{\frac{k}{2}} \times \left[\frac{(n-k-1)w}{k(1-w)}\right]^{\frac{k}{2}-1}\left[1+\frac{k}{n-k-1}\frac{(n-k-1)}{k}\frac{w}{1-w}\right]^{-\frac{n-1}{2}}$$

$$= \frac{(n-k-1)k(w+1-w)}{k^2(1-w)^2}\cdot\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{k}{2})\cdot\Gamma(\frac{n-k-1}{2})}\cdot\left[\frac{k}{n-k-1}\cdot\frac{n-k-1}{k}\right]^{k/2}\cdot\frac{k}{n-k-1}\left(\frac{w}{1-w}\right)^{\frac{k}{2}-1}\left(\frac{1}{1-w}\right)^{-\frac{n-1}{2}}$$

$$= \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k-1}{2})}\cdot w^{\frac{k}{2}-1}\cdot(1-w)^{\frac{n-k-1}{2}-1}$$

$$= \frac{w^{\frac{k}{2}-1}(1-w)^{\frac{n-k-1}{2}-1}}{B(\frac{k}{2}+\frac{n-k-1}{2})} \sim B\left(\frac{k}{2}, \frac{n-k-1}{2}\right)$$

$$\therefore R^2 \sim B\left(\frac{k}{2}, \frac{n-k-1}{2}\right)$$