Homework 10 Solutions

Exercise 1:
The call is exercised if \( S_T > E \)
or \( \ln S_T > \ln E \)

But \( \ln S_T \sim N \left( \ln S_0 + (r - \frac{\sigma^2}{2})T, \sigma \sqrt{T} \right) \)

\[
p \left( \ln S_T > \ln E \right) = p \left( Z > \frac{\ln E - \ln S_0 - (r - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right)
\]

\[
= p \left( Z < \frac{-\ln E + \ln S_0 + (r - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right)
\]

\[
= p \left( Z < \frac{\ln \frac{S_0}{E} + (r - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right) = \phi \left( \frac{36 - 30 e^{0.06 \times \frac{6}{12}}}{0.06 \times \frac{6}{12}} \right)
\]

Exercise 2:

\[
\begin{align*}
30 & \quad \text{P} = 36 + (1 - \text{P})27 \\
27 & \quad \text{P} = 30 e^{0.06 \times \frac{6}{12}} - 27
\end{align*}
\]

\[
P = 0.4348
\]
**EXERCISE 3:**

\[ D_1 = 1.50 \quad D_2 = 1.50 \]

\[ t_0 = 0 \quad t_1 = 4 \quad t_2 = 10 \quad T = 15 \]

Check if \( D_2 < \mathbb{E}(1 - e^{-r(T-t_2)}) = 55(1 - e^{-0.08 \frac{5}{12}}) = 1.80 > D_2 \)

and \( D_1 < \mathbb{E}(1 - e^{-r(T-t_1)}) = 55(1 - e^{-0.08 \frac{7}{12}}) = 2.15 > D_1 \)

Therefore it will not be optimal to exercise early.

To find the price of the call we use the dividend discount model to find \( S_0^* = S_0 - \text{PV}(D_1) - \text{PV}(D_2) \)

And then apply the Black-Scholes-Merton model.

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**EXERCISE 4:**

(a): \( C = S \cdot \frac{2Y}{\sigma^2} \)

\[ \frac{dC}{dt} = 0, \quad \frac{dC}{dS} = -\frac{2Y}{\sigma^2} \cdot S \]

\[ \frac{d^2C}{dS^2} = -\frac{2Y}{\sigma^2} \left( \frac{-2Y}{\sigma^2} - 1 \right) \cdot S \]

**BLACK-SCHOLES-MERTON DIFFERENTIAL EQUATION:**

\[ \frac{dC}{dt} + \frac{1}{2} \sigma^2 \frac{d^2C}{dS^2} + rS \frac{dC}{dS} - rC = 0 \]

Yes, it satisfies the equation.
\[(b), \text{ Payoff at Expiration } = \max \left( S_{0}e^{rt} - E, 0 \right) \]

\[ \therefore \text{ The price of the call option is} \]

\[ e^{-rt} \max \left( S_{0}e^{rt} - E, 0 \right) = \max \left( S_{0} - Ee^{-rt}, 0 \right) \]

\[(c), \text{ If } S_{0} > Ee^{-rt} \rightarrow m_{1}S_{0} - m_{1}E + rt > 0 \]

or \[ m_{1} \left( \frac{S_{0}}{E} \right) + rt > 0 \]

\[ \therefore d_{1} \rightarrow +\infty \text{ and } d_{2} \rightarrow +\infty \text{ as } \sigma \rightarrow 0 \]

\[ \therefore \phi(d_{1}) = 1 \]

\[ \therefore \phi(d_{2}) = 1 \]

\[ \therefore C = S_{0} - Ee^{-rt} \]

\[ \text{If } S_{0} < Ee^{-rt} \rightarrow m_{1}S_{0} - m_{1}E + rt < 0 \]

or \[ m_{1} \left( \frac{S_{0}}{E} \right) + rt < 0 \]

\[ \therefore d_{1} \rightarrow -\infty \text{ and } d_{2} \rightarrow -\infty \text{ as } \sigma \rightarrow 0 \]

\[ \therefore \phi(d_{1}) = 0 \]

\[ \therefore \phi(d_{2}) = 0 \]

\[ \therefore C = 0. \]
**Exercise 5:**

(a) \[ D_n \leq e(1-e) \]

Using the series expansion of \( e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \)

we get \( e^x = 1 - \frac{x(T-t)}{1!} + \frac{x^2(T-t)^2}{2!} + \ldots \)

Thus \[ D_n \leq e(T-t) \]

(b) Use Jensen's Inequality

Suppose the line is tangent at \( m_{CST} \)

\( m_{ST} \leq a + bST \) because \( m_{ST} \) is concave

\( E[m_{ST}] \leq a + bE[ST] \)

\( E[m_{ST}] \leq m[CST] \)
(c). ASSSET A: 1% of $1,000,000 = $1000
ASSSET B: 1% of $1,000,000 = $1000

\[ \rho = 0.3 \]

\[ \text{SD} (X+Y) = \sqrt{1000^2 + 1000^2 + 2(0.3)(1000)(1000)} = 1612.5 \]

\[ \therefore \text{1-day 95% VAR} = 1.645 (1612.5) = 2652.6 \]

\[ \text{AND 5-day 95% VAR} = 2652.6 \sqrt{5} = 5931.4. \]

OR USING THE LINEAR MODEL:

\[ \Delta P = \sum \Delta X_i \]

\[ \text{VAR}(\Delta P) = \begin{bmatrix} 100000 & 100000 \\ 0.3(0.1) & 0.3(0.1) \end{bmatrix} \begin{bmatrix} 100000 \\ 0.3(0.1) \end{bmatrix} = 26000000 \]

\[ \text{SD}(\Delta P) = \sqrt{26000000} = 1612.5. \]