HOMEWORK 10 SOLUTIONS

EXERCISE 1:

THE CALL IS EXERCISED IF $S_T > E$

OR $\ln S_T > \ln E$

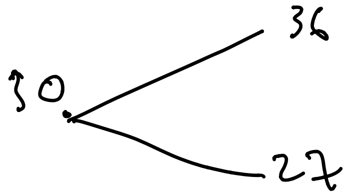
BUT $\ln S_T \sim N\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right)$

$$\therefore P\left(\ln S_T > \ln E\right) = P\left(Z > \frac{\ln E - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= P\left(Z < \frac{-\ln E + \ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= P\left(Z < \frac{\ln \frac{S_0}{E} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = \Phi(d_2).$$

EXERCISE 2:



$$30\, e^{0.06 \frac{6}{12}} = p\,36 + (1-p)27$$

$$9p = 30\, e^{0.06 \frac{6}{12}} - 27$$

$$p = 0.4348$$

EXERCISE 3:

$D_1 = 1.50$ $\quad$ $D_2 = 1.50$

$t = 0$ $\quad$ $t_1 = 4$ $\quad$ $t_2 = 10$ $\quad$ $T = 15$

CHECK IF $D_2 < E\left(1 - e^{-r(T - t_2)}\right) = 55\left(1 - e^{-0.08 \frac{5}{12}}\right) = 1.80 > D_2$

AND $D_1 < E\left(1 - e^{-r(t_2 - t_1)}\right) = 55\left(1 - e^{-0.08 \frac{6}{12}}\right) = 2.15 > D_1$

THEREFORE IT WILL NOT BE OPTIMAL TO EXERCISE EARLY.
TO FIND THE PRICE OF THE CALL USE THE DIVIDEND DISCOUNT MODEL TO FIND $S_0' = S_0 - PV(D_1) - PV(D_2)$
AND THEN APPLY THE BLACK-SCHOLES-MERTON MODEL.

EXERCISE 4:

(a). $C = S^{-\frac{2r}{\sigma^2}}$

$$\frac{\partial C}{\partial t} = 0, \quad \frac{\partial C}{\partial S} = -\frac{2r}{\sigma^2} S^{-\frac{2r}{\sigma^2} - 1}$$

$$\frac{\partial^2 C}{\partial S^2} = -\frac{2r}{\sigma^2}\left(-\frac{2r}{\sigma^2} - 1\right) S^{-\frac{2r}{\sigma^2} - 2}$$

BLACK-SCHOLES-MERTON DIFFERENTIAL EQUATION:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + rS\frac{\partial C}{\partial S} - rC = 0$$

YES, IT SATISFIES THE EQUATION.

(b). PAYOFF AT EXPIRATION $= \text{MAX}\left(S_0 e^{rt} - E, 0\right)$

$\therefore$ THE PRICE OF THE CALL OPTION IS

$$e^{-rt}\,\text{MAX}\left(S_0 e^{rt} - E, 0\right) = \text{MAX}\left(S_0 - E e^{-rt}, 0\right)$$

(c). IF $S_0 > E e^{-rt} \rightarrow \ln S_0 - \ln E + rt > 0$

OR $\ln\left(\frac{S_0}{E}\right) + rt > 0$

$\therefore$ $d_1 \rightarrow +\infty$ AS $\sigma \rightarrow 0$ AND $d_2 \rightarrow +\infty$ AS $\sigma \rightarrow 0$

$\hookrightarrow \phi(d_1) = 1$ $\quad$ $\hookrightarrow \phi(d_2) = 1$

$\therefore$ $C = S_0 - E e^{-rt}$.

IF $S_0 < E e^{-rt} \rightarrow \ln S_0 - \ln E + rt < 0$

OR $\ln\left(\frac{S_0}{E}\right) + rt < 0$

$\therefore$ $d_1 \rightarrow -\infty$ AS $\sigma \rightarrow 0$ AND $d_2 \rightarrow -\infty$ AS $\sigma \rightarrow 0$

$\hookrightarrow \phi(d_1) = 0$ $\quad$ $\hookrightarrow \phi(d_2) = 0$

$\therefore$ $C = 0$.

Exercise 5:

Black-Scholes-Merton model: $S_0 = \$95 \qquad E = \$105 \qquad \sigma = 0.60 \qquad t = \frac{2}{3}$

$$
\begin{aligned}
C &= S_0\Phi(d_1) - \left(\frac{E}{e^{rt}}\right)\Phi(d_2) \\
d_1 &= \frac{ln\left(\frac{S_0}{E}\right) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = \frac{ln\left(\frac{95}{105}\right) + (0.08 + \frac{1}{2}0.60^2)\frac{2}{3}}{0.60\sqrt{\frac{2}{3}}} = 0.149521 \\
d_2 &= \frac{ln\left(\frac{S_0}{E}\right) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t} = 0.149521 - 0.489898 = -0.340377 \\
\Phi(d_1) &= \Phi(0.149521) = 0.5594287 \\
\Phi(d_2) &= \Phi(-0.340377) = 0.3667863 \\
C &= 95 \times 0.5594287 - \left(\frac{105}{e^{0.08\times\frac{2}{3}}}\right)0.3667863 = 16.6334
\end{aligned}
$$