REFERENCES

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Capital Market Equilibrium with Restricted Borrowing†

INTRODUCTION

Several authors have contributed to the development of a model describing the pricing of capital assets under conditions of market equilibrium.[1] The model states that under certain assumptions the expected return on any capital asset for a single period will satisfy

$$E(\tilde{R}_i) = R_f + \beta_i[E(\tilde{R}_m) - R_f].$$ (1)

The symbols in equation (1) are defined as follows: $\tilde{R}_i$ is the return on asset i for the period and is equal to the change in the price of the asset, plus any dividends, interest, or other distributions, divided by the price of the asset at the start of the period; $\tilde{R}_m$ is the return on the market portfolio of all assets taken together; R_f is the return on a riskless asset for the period; β_i is the "market sensitivity" of asset i and is equal to the slope of the regression line relating $\tilde{R}_i$ and $\tilde{R}_m$. The market sensitivity β_i of asset i is defined algebraically by

$$\beta_i = \mathrm{cov}(\tilde{R}_i, \tilde{R}_m)/\mathrm{var}(\tilde{R}_m).$$ (2)

The assumptions that are generally used in deriving equation (1) are as follows: (a) All investors have the same opinions about the possibilities of various end-of-period values for all assets. They have a common joint probability distribution for the returns on the available assets. (b) The common probability distribution describing the possible returns on the available assets is joint normal (or joint stable with a single characteristic exponent). (c) Investors choose portfolios that maximize their expected end-of-period utility of wealth, and all investors are risk averse. (Every investor's utility function on end-of-period wealth increases at a decreasing rate as his wealth increases.) (d) An investor may take a long or short position of any size in any asset, including the riskless asset. Any investor may borrow or lend any amount he wants at the riskless rate of interest.

The length of the period for which the model applies is not specified. The assumptions of the model make sense, however, only if the period is taken to be infinitesimal. For any finite period, the distribution of possible returns on an asset is likely to be closer to lognormal than normal;

1. A summary of the development of the model may be found in William F. Sharpe, *Portfolio Theory and Capital Markets* (New York: McGraw-Hill Book Co., 1970).

444

in particular, if the distribution of returns is normal, then there will be a finite probability that the asset will have a negative value at the end of the period.

Of these assumptions, the one that has been felt to be the most restrictive is (d). Lintner has shown that removing assumption (a) does not change the structure of capital asset prices in any significant way,[2] and assumptions (b) and (c) are generally regarded as acceptable approximations to reality. Assumption (d), however, is not a very good approximation for many investors, and one feels that the model would be changed substantially if this assumption were dropped.

In addition, several recent studies have suggested that the returns on securities do not behave as the simple capital asset pricing model described above predicts they should. Pratt analyzes the relation between risk and return in common stocks in the 1926–60 period and concludes that high-risk stocks do not give the extra returns that the theory predicts they should give.[3] Friend and Blume use a cross-sectional regression between risk-adjusted performance and risk for the 1960–68 period and observe that high-risk portfolios seem to have poor performance, while low-risk portfolios have good performance.[4] They note that there is some bias in their test, but claim alternately that the bias is so small that it can be ignored, and that it explains half of the effect they observe.[5] In fact, the bias is serious. Miller and Scholes do an extensive analysis of the nature of the bias and make corrections for it.[6] Even after their corrections, however, there is a negative relation between risk and performance.

Black, Jensen, and Scholes analyze the returns on portfolios of stocks at different levels of β_i in the 1926–66 period.[7] They find that the average returns on these portfolios are not consistent with equation (1), especially in the postwar period 1946–66. Their estimates of the expected returns on portfolios of stocks at low levels of β_i are consistently higher than predicted by equation (1), and their estimates of the expected returns on portfolios of stocks at high levels of β_i are consistently lower than predicted by equation (1).

2. John Lintner, "The Aggregation of Investors' Diverse Judgments and Preferences in Perfectly Competitive Security Markets," *Journal of Financial and Quantitative Analysis* 4 (December 1969): 347–400.

3. Shannon P. Pratt, "Relationship between Viability of Past Returns and Levels of Future Returns for Common Stocks, 1926–1960," memorandum (April 1967).

4. Irwin Friend and Marshall Blume, "Measurement of Portfolio Performance under Uncertainty," *American Economic Review* 60 (September 1970): 561–75.

5. Ibid., p. 568. Compare the text with n. 15.

6. Merton H. Miller and Myron Scholes, "Rates of Return in Relation to Risk: A Re-Examination of Some Recent Findings," in *Studies in the Theory of Capital Markets,* ed. Michael C. Jensen (New York: Praeger Publishing Co., in press).

7. Fischer Black, Michael C. Jensen, and Myron Scholes, "The Capital Asset Pricing Model: Some Empirical Tests," in *Studies in the Theory of Capital Markets,* ed. Michael C. Jensen (New York: Praeger Publishing Co., in press).

Black, Jensen, and Scholes also find that the behavior of well-diversified portfolios at different levels of β_i is explained to a much greater extent by a "two-factor model" than by a single-factor "market model."[8] They show that a model of the following form provides a good fit for the behavior of these portfolios:

$$\tilde{R}_i = a_i + b_i \tilde{R}_m + (1 - b_i) \tilde{R}_z + \tilde{\epsilon}_i. \tag{3}$$

In equation (3), $\tilde{R}_z$ is the return on a "second factor" that is independent of the market (its β_i is zero), and $\tilde{\epsilon}_i$, $i = 1, 2, \ldots, N$ are approximately mutually independent residuals.

This model suggests that in periods when R_z is positive, the low β_i portfolios all do better than predicted by equation (1), and the high β_i portfolios all do worse than predicted by equation (1). In periods when R_z is negative, the reverse is true: low β_i portfolios do worse than expected, and high β_i portfolios do better than expected. In the postwar period, the estimates obtained by Black, Jensen, and Scholes for the mean of $\tilde{R}_z$ were significantly greater than zero.

One possible explanation for these empirical results is that assumption (d) of the capital asset pricing model does not hold. What we will show below is that the relaxation of assumption (d) can give models that are consistent with the empirical results obtained by Pratt, Friend and Blume, Miller and Scholes, and Black, Jensen and Scholes.

EQUILIBRIUM WITH NO RISKLESS ASSET

Let us start by assuming that investors may take long or short positions of any size in any risky asset, but that there is no riskless asset and that no borrowing or lending at the riskless rate of interest is allowed. This assumption is not realistic, since restrictions on short selling are at least as stringent as restrictions on borrowing. But restrictions on short selling may simply add to the effects that we will show are caused by restrictions on borrowing. Under these assumptions, Sharpe shows that the efficient set of portfolios may be written as a weighted combination of two basic portfolios, with different weights being used to generate the different portfolios in the efficient set.[9] In his notation, the proportion X_i of asset i in the efficient portfolio corresponding to the parameter λ satisfies (4), where K_i and k_i are constants:

$$X_i = K_i + \lambda k_i \qquad i = 1, 2, \ldots, N. \tag{4}$$

Thus the weights on the stocks in the two basic portfolios are K_i, $i = 1$, $2, \ldots, N$, and k_i, $i = 1, 2, \ldots, N$. The weights satisfy (5), so the sum of the weights X_i is always equal to 1.

8. One form of market model is defined in Eugene F. Fama, "Risk, Return, and Equilibrium," *Journal of Political Economy* 79 (January/February 1971): 34.
9. Sharpe, pp. 59–69.

$$\sum_{i=1}^{N} K_i = 1; \qquad \sum_{i=1}^{N} k_i = 0. \tag{5}$$

Sharpe also shows that the variance of return on an efficient portfolio is a quadratic function of its expected return.

Similarly, Lintner shows that a number of relations can be derived when there is no riskless asset.[10] His equation (16c) can be interpreted, in the case where all investors agree on the joint distribution of end-of-period values for all assets, as saying that even when there is no riskless asset, every investor holds a linear combination of two basic portfolios. And his equation (18) can be interpreted as saying that the prices of assets in equilibrium are related in a relatively simple way even without a riskless asset.

Cass and Stiglitz show that if the returns on securities are not assumed to be joint normal, but are allowed to be arbitrary, then the set of efficient portfolios can be written as a weighted combination of two basic portfolios only for a very special class of utility functions.[11]

Using a notation similar to that used by Fama, we can show that every efficient portfolio consists of a weighted combination of two basic portfolios as follows. An efficient portfolio is one that has maximum expected return for given variance, or minimum variance for given expected return. Thus the efficient portfolio held by individual k is obtained by choosing proportions x_{ki}, $i = 1, 2, \ldots, N$, invested in the shares of each of the N available assets, in order to

Minimize: $\qquad \operatorname{var}(\widetilde{R}_k) = \sum_{i=1}^{N} \sum_{j=1}^{N} x_{ki} x_{kj} \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_j);$ $\qquad$ (6)

Subject to: $\qquad E(\widetilde{R}_k) = \sum_{j=1}^{N} x_{kj} E(\widetilde{R}_j);$ $\qquad$ (7)

$$\sum_{j=1}^{N} x_{kj} = 1. \tag{8}$$

Using Lagrange multipliers S_k and T_k, this can be expressed as

Minimize: $\qquad \sum_{i=1}^{N} \sum_{j=1}^{N} x_{ki} x_{kj} \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_j)$

(9)

$$- 2S_k \left[\sum_{j=1}^{N} x_{kj} E(\widetilde{R}_j) - E(\widetilde{R}_k) \right] - 2T_k \left[\sum_{j=1}^{N} x_{kj} - 1 \right].$$

10. Lintner, pp. 373–84.

11. David Cass and Joseph E. Stiglitz, "The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds," *Journal of Economic Theory* 2 (June 1970): 122–60.

Taking the derivative of this expression with respect to x_{ki}, we have

$$\sum_{j=1}^{N} x_{kj} \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_j) - S_k E(\widetilde{R}_i) - T_k = 0. \tag{10}$$

This set of equations, for $i = 1, 2, \ldots, N$, determines the values of x_{ki}. If we write D_{ij} for the inverse of the covariance matrix $\operatorname{cov}(\widetilde{R}_i, \widetilde{R}_j)$, then the solution to this set of equations may be written

$$x_{ki} = S_k \sum_{j=1}^{N} D_{ij} E(\widetilde{R}_j) + T_k \sum_{j=1}^{N} D_{ij}. \tag{11}$$

Note that the subscript k, referring to the individual investor, appears on the right-hand side of this equation only in the multipliers S_k and T_k. Thus every investor holds a linear combination of two basic portfolios, and every efficient portfolio is a linear combination of these two basic portfolios. In equation (11), there is no guarantee that the weights on the individual assets in the two portfolios sum to one. If we normalize these weights, then equation (11) may be written

$$x_{ki} = w_{kp} x_{pi} + w_{kq} x_{qi}. \tag{12}$$

In equation (12), the symbols are defined as follows:

$$\begin{aligned} w_{kp} &= S_k \sum_{i=1}^{N} \sum_{j=1}^{N} D_{ij} E(\widetilde{R}_j); \\ w_{kq} &= T_k \sum_{i=1}^{N} \sum_{j=1}^{N} D_{ij}; \\ x_{pi} &= \sum_{j=1}^{N} D_{ij} E(\widetilde{R}_j) \Big/ \sum_{i=1}^{N} \sum_{j=1}^{N} D_{ij} E(\widetilde{R}_j); \\ x_{qi} &= \sum_{j=1}^{N} D_{ij} \Big/ \sum_{i=1}^{N} \sum_{j=1}^{N} D_{ij}. \end{aligned} \tag{13}$$

Thus we have

$$\begin{aligned} \sum_{i=1}^{N} x_{pi} &= 1; \\ \sum_{i=1}^{N} x_{qi} &= 1; \\ w_{kp} + w_{kq} &= 1 \qquad k = 1, 2, \ldots, L. \end{aligned} \tag{14}$$

The last equation in (14) follows from the fact that the x_{ki}'s must also sum to one.

Equation (12), then, shows that the efficient portfolio held by investor k consists of a weighted combination of the basic portfolios p and q. Note, however, that the two basic portfolios are not unique. Suppose that we transform the basic portfolios p and q into two different portfolios u and v, using weights w_{up}, w_{uq}, w_{vp}, and w_{vq}. Then we have

$$x_{ui} = w_{up}x_{pi} + w_{uq}x_{qi};$$

$$x_{vi} = w_{vp}x_{pi} + w_{vq}x_{qi}. \tag{15}$$

Normally, we will be able to solve equations (15) for x_{pi} and x_{qi}. Let us write the resulting coefficients w_{pu}, w_{pv}, w_{qu}, and w_{qv}. Then we will have

$$x_{pi} = w_{pu}x_{ui} + w_{pv}x_{vi};$$

$$x_{qi} = w_{qu}x_{ui} + w_{qv}x_{vi}. \tag{16}$$

Substituting equations (16) into equation (12), we see that we can write the efficient portfolio k as a linear combination of the new basic portfolios u and v as follows:

$$x_{ki} = w_{ku}x_{ui} + w_{kv}x_{vi}. \tag{17}$$

In equation (17), the weights w_{ku} and w_{kv} sum to one.

Thus the basic portfolios u and v can be any pair of different portfolios that can be formed as weighted combinations of the original pair of basic portfolios p and q. Every efficient portfolio can be expressed as a weighted combination of portfolios u and v, but they need not be efficient themselves.

Portfolios p and q must have different β's, if it is to be possible to generate every efficient portfolio as a weighted combination of these two portfolios. But if they have different β's, then it will be possible to generate new basic portfolios u and v with arbitrary β's, by choosing appropriate weights. In particular, let us choose weights such that

$$\beta_u = 1; \qquad \beta_v = 0. \tag{18}$$

Multiplying equation (12) by the fraction x_{mk} of total wealth held by investor k, and summing over all investors ($k = 1, 2, \ldots, L$), we obtain the weights x_{mi} of each asset in the market portfolio:

$$x_{mi} = \left(\sum_{k=1}^{L} x_{mk}w_{kp} \right) x_{pi} + \left(\sum_{k=1}^{L} x_{mk}w_{kq} \right) x_{qi}. \tag{19}$$

Since the market portfolio is a weighted combination of portfolios p and q, and since β_m is one, portfolio u must be the market portfolio. Thus we can rename the portfolios u and v specified by (18) portfolios m and z, for the market portfolio and the zero-β basic portfolio. When we write the return on an efficient portfolio k as a weighted combination of the returns on portfolios m and z, the coefficient of the return on portfolio m must be β_k. Thus we can write

$$\tilde{R}_k = \beta_k \tilde{R}_m + (1 - \beta_k) \tilde{R}_z. \qquad (20)$$

Taking expected values of both sides of equation (20), and rewriting slightly, we have

$$E(\tilde{R}_k) = E(\tilde{R}_z) + \beta_k[E(\tilde{R}_m) - E(\tilde{R}_z)]. \qquad (21)$$

Equation (21) says that the expected return on an efficient portfolio k is a linear function of its β_k. From (1), we see that the corresponding relationship when there is a riskless asset and riskless borrowing and lending are allowed is

$$E(\tilde{R}_k) = R_f + \beta_k[E(\tilde{R}_m) - R_f]. \qquad (22)$$

Thus the relation between the expected return on an efficient portfolio k and its β_k is the same whether or not there is a riskless asset. If there is, then the intercept of the relationship is R_f. If there is not, then the intercept is $E(\tilde{R}_z)$.

We can now show that equation (21) applies to individual securities as well as to efficient portfolios. Subtracting equation (10) from itself after permuting the indexes, we get

$$\text{cov}(\tilde{R}_i, \tilde{R}_k) - \text{cov}(\tilde{R}_j, \tilde{R}_k) = S_k[E(\tilde{R}_i) - E(\tilde{R}_j)]. \qquad (23)$$

Since the market is an efficient portfolio, we can put m for k, and since i and j can be taken to be portfolios as well as assets, we can put z for j. Then equation (23) becomes

$$\text{cov}(\tilde{R}_i, \tilde{R}_m) = S_m[E(\tilde{R}_i) - E(\tilde{R}_z)]. \qquad (24)$$

Equation (24) may be rewritten as

$$E(\tilde{R}_i) = E(\tilde{R}_z) + [\text{var}(\tilde{R}_m)/S_m]\beta_i. \qquad (25)$$

Putting m for i in equation (25), we find

$$\text{var}(\tilde{R}_m)/S_m = E(\tilde{R}_m) - E(\tilde{R}_z). \qquad (26)$$

So equation (25) becomes

$$E(\tilde{R}_i) = E(\tilde{R}_z) + \beta_i[E(\tilde{R}_m) - E(\tilde{R}_z)]. \qquad (27)$$

Thus the expected return on every asset, even when there is no riskless asset and riskless borrowing is not allowed, is a linear function of its β. Comparing equation (27) with equation (1), we see that the introduction of a riskless asset simply replaces $E(\tilde{R}_z)$ with R_f.

Now we can derive another property of portfolio z. Equation (27) holds for any asset and thus for any portfolio. Setting $\beta_i = 0$, we see that every portfolio with β equal to zero must have the same expected return as portfolio z. Since the return on portfolio z is independent of the return on portfolio m, and since weighted combinations of portfolios m and z must be efficient, portfolio z must be the minimum-variance zero-β portfolio.

Fama comes close to deriving equation (27). His equation (27) says that the expected return on an asset is a linear function of its risk, measured relative to an efficient portfolio containing the asset. Lintner also derives a linear relationship (eq. [18]) between the expected return on an asset and its risk. It is possible to derive equation (27) from either Fama's or Lintner's equations in a relatively small number of steps.

Fama, however, goes on to introduce the concept of a new kind of financial intermediary that he calls a "portfolio sharing company." In the absence of riskless borrowing or lending opportunities, he says that this fund can purchase units of the market portfolio, and sell shares in its return to different investors. He says that an investor can specify the proportion of the return on this fund that he will receive *per unit of his own funds invested.* Writing β_k for this proportion, Fama claims that

$$\tilde{R}_k = \beta_k \tilde{R}_m. \tag{28}$$

But this is not consistent with market equilibrium. Assuming that $E(R_z)$ is positive, shares in this fund will be less attractive than direct holdings of efficient portfolios with β_k less than one, as given by equation (20). If $E(\tilde{R}_z)$ is negative, shares in this fund will be less attractive than direct holdings of efficient portfolios with β_k greater than one. So there is no way that the fund can sell all of its shares, except, of course, that it can determine a number R_f such that when the return on the holdings of investor k is defined by equation (29), all of the fund's shares can be sold:

$$\tilde{R}_k = R_f + \beta_k(\tilde{R}_m - R_f). \tag{29}$$

But this is just an implicit way of creating borrowing and lending opportunities. So the concept of portfolio sharing does not cast any light on market equilibrium in the absence of riskless borrowing and lending opportunities.

Starting with equation (23), we can now show one final property of portfolio z. Let p and q be two efficient portfolios and let w_{zp} and w_{zq} be the weights that give portfolio z when applied to portfolios p and q. Putting m for j and p for k to give one equation, and putting m for j and q for k to give another, we have

$$\text{cov}(\tilde{R}_i, \tilde{R}_p) - \text{cov}(\tilde{R}_m, \tilde{R}_p) = S_p[E(\tilde{R}_i) - E(\tilde{R}_m)];$$
$$\text{cov}(\tilde{R}_i, \tilde{R}_q) - \text{cov}(\tilde{R}_m, \tilde{R}_q) = S_q[E(\tilde{R}_i) - E(\tilde{R}_m)]. \tag{30}$$

Multiplying the equations by w_{zp} and w_{zq}, respectively, and adding them —noting that $\text{cov}(\tilde{R}_m, \tilde{R}_z)$ is zero—we have

$$\text{cov}(\tilde{R}_i, \tilde{R}_z) = (w_{zp}S_p + w_{zq}S_q)[E(\tilde{R}_i) - E(\tilde{R}_m)]. \tag{31}$$

Substituting for $E(\tilde{R}_i)$ from equation (27), we obtain

$$\text{cov}(\tilde{R}_i, \tilde{R}_z) = (1 - \beta_i)(w_{zp}S_p + w_{zq}S_q)[E(\tilde{R}_z) - E(\tilde{R}_m)]. \tag{32}$$

Thus we see that the covariance of the return on any asset i with the return on portfolio z is proportional to $1 - \beta_i$.

In sum, we have shown that when there is no riskless asset, and no riskless borrowing or lending, every efficient portfolio may be written as a weighted combination of the market portfolio m and the minimum-variance zero-β portfolio z. The covariance of the return on any asset i with the return on portfolio z is proportional to $1 - \beta_i$. The expected return on any asset or portfolio i depends only on β_i, and is a linear function of β_i.

Prohibition of borrowing and lending, then, shifts the intercept of the line relating $E(R_i)$ and β_i from R_f to $E(\tilde{R}_z)$. Since this is the effect that complete prohibition would have, it seems likely that partial restrictions on borrowing and lending, such as margin requirements, would also shift the intercept of the line, but less so. Thus it is possible that restrictions on borrowing and lending would lead to a market equilibrium consistent with the empirical model expressed in equation (3) and developed by Black, Jensen, and Scholes.

EQUILIBRIUM WITH NO RISKLESS BORROWING

Let us turn now to the case in which there is a riskless asset available, such as a short-term government security, but in which investors are not allowed to take short positions in the riskless asset. We will continue to assume that investors may take short positions in risky assets.

Vasicek has shown that in this case the principal features of the equilibrium with no riskless borrowing or lending are preserved.[12] The expected return on any asset i continues to be a function only of its β. The function is still linear. The efficient set of portfolios now has two parts, however. One part consists of weighted combinations of portfolios m and z, and the other part consists of weighted combinations of the riskless asset with a single portfolio of risky assets that we can call portfolio t.

We can show this, in our notation, as follows. Since the restriction on borrowing applies only to the riskless asset, there will be only two kinds of efficient portfolios, those that contain the riskless asset and those that do not. Let us call the riskless asset number $N + 1$.

For those efficient portfolios that do not contain the riskless asset, equations (6)–(18) of the previous section apply. Each such efficient portfolio can be expressed as a weighted combination of portfolios u and v, where β_u is one and β_v is zero.

For those efficient portfolios that do contain the riskless asset, we can extend equation (10) to $N + 1$ assets. The covariance term for $j = N + 1$ vanishes, so we have

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12. Oldrich A. Vasicek, "Capital Market Equilibrium with No Riskless Borrowing," memorandum (March 1971); available from the Wells Fargo Bank.

$$\sum_{j=1}^{N} x_{kj} \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_j) - S_k E(\widetilde{R}_i) - T_k = 0. \qquad (33)$$

For $i = 1, 2, \ldots, N$, this set of equations determines values for x_{ki}, $i = 1, 2, \ldots, N$, as before. Thus we see that the risky portions of these investors' portfolios are weighted combinations of portfolios u and v. For $i = N + 1$, equation (33) becomes

$$-S_k R_f - T_k = 0 \qquad k = 1, 2, \ldots, L. \qquad (34)$$

This means that every investor places the same relative weights S_k and T_k on portfolios u and v. Let us write t for the portfolio of risky assets containing relative weights S_k and T_k of portfolios u and v. Then every investor who holds the riskless asset holds a weighted combination of the riskless asset and portfolio t.

Since the risky part of every investor's portfolio, whether he holds the riskless asset or not, consists of a weighted combination of portfolios u and v, the sum of all investors' risky holdings, which is the market portfolio, must be a weighted combination of portfolios u and v. Using arguments parallel to those used in the last section, we can show that portfolio u must be the market portfolio, and portfolio v must be the minimum-variance zero-β portfolio of risky assets.

Equation (33) is the same as equation (10), so we can see that it holds for all risky assets i, $i = 1, 2, \ldots, N$, and all efficient portfolios k. Equations (23)–(27) go through as before, and we see that equation (27) applies to all risky assets even when there are riskless lending opportunities.

Now we can derive some additional properties of portfolios z and t. Let us write w_{km}, w_{kz}, and w_{kf} for the weights on portfolios m, z, and the riskless asset in efficient portfolio k. Since the return on portfolio z is independent of the return on portfolio m, the expected return and variance of portfolio k will be

$$E(\widetilde{R}_k) = w_{km} E(\widetilde{R}_m) + w_{kz} E(\widetilde{R}_z) + w_{kf} R_f; \qquad (35)$$

$$\operatorname{var}(\widetilde{R}_k) = w_{km}{}^2 \operatorname{var}(\widetilde{R}_m) + w_{kz}{}^2 \operatorname{var}(\widetilde{R}_z). \qquad (36)$$

The weights must also satisfy constraints (37) and (38):

$$w_{km} + w_{kz} + w_{kf} = 1; \qquad (37)$$

$$w_{kf} \geqslant 0. \qquad (38)$$

We can see immediately that $E(\widetilde{R}_z)$ must satisfy

$$R_f < E(\widetilde{R}_z) < E(\widetilde{R}_m). \qquad (39)$$

If $E(\widetilde{R}_z)$ is less than or equal to R_f, then we can increase w_{kf} and decrease w_{kz} by the same amount, and we will reduce the variance of portfolio k and increase or leave unchanged its expected return. But if this is possible, it means that portfolio k is not efficient.

When portfolio k is the market portfolio, w_{km} must be one, and w_{kz} must be zero. If $E(\widetilde{R}_z)$ is greater than or equal to $E(\widetilde{R}_m)$, then we can decrease w_{km} by a very small amount and increase w_{kz} by the same amount, and we will reduce the variance of portfolio k and increase or leave unchanged its expected return. But if this is possible, it means that the market portfolio is not efficient. Thus the inequality (39) must hold.

When w_{kf} is greater than zero, portfolio k is a mixture of portfolio t and the riskless asset. We can incorporate equation (37) in equation (35) as follows:

$$E(\widetilde{R}_k - R_f) = w_{km}[E(\widetilde{R}_m - R_f)] + w_{kz}[E(\widetilde{R}_z - R_f)]. \quad (40)$$

Equation (36) may be written equivalently as

$$\operatorname{var}(\widetilde{R}_k - R_f) = w_{km}{}^2 \operatorname{var}(\widetilde{R}_m - R_f) + w_{kz}{}^2 \operatorname{var}(\widetilde{R}_z - R_f). \quad (41)$$

Since equations (40) and (41) hold for any portfolio containing the riskless asset, they must hold also for portfolio t. Since portfolio t is efficient, it must maximize (40) subject to (41). But the solution to that problem is the same as the solution to

Maximize: $\qquad E(\widetilde{R}_k - R_f)/\sigma(\widetilde{R}_k - R_f).$ $\qquad\qquad$ (42)

But when the efficient portfolios are plotted on a graph with $E(\widetilde{R}_k - R_f)$ on the y-axis, and $\sigma(\widetilde{R}_k - R_f)$—which is the same as $\sigma(\widetilde{R}_k)$—on the x-axis, the value of k that satisfies (42) is the value of k that maximizes the slope of a line drawn from the origin to point k. So portfolio t is the "tangent portfolio" to the efficient set.

In sum, the introduction of riskless lending opportunities changes the nature of the market equilibrium in just one way. There are now two kinds of efficient portfolios. The less risky efficient portfolios are mixtures of portfolio t and the riskless asset. The more risky efficient portfolios continue to be mixtures of portfolios m and z. Portfolio t itself is a mixture of portfolios m and z. The expected return on portfolio z must now be greater than the return on the riskless asset. The expected return on a security continues to be a linear function of its β.

Thus the empirical results reported by Black, Jensen, and Scholes are consistent with a market equilibrium in which there are riskless lending opportunities, as well as with an equilibrium in which there are no riskless borrowing or lending opportunities. The general approach used in this section can be used to obtain similar results when every individual has a limit on the amount he can borrow that may be greater than zero. Thus we can say that the empirical results are consistent with an equilibrium in which borrowing at the riskless interest rate is either fully or partially restricted.

CONCLUSIONS

We have explored the nature of capital market equilibrium under two assumptions that are more restrictive than the usual assumptions used in

deriving the capital asset pricing model. First, we have assumed that there is no riskless asset and that no riskless borrowing or lending is allowed. Then we have assumed that there is a riskless asset and that long positions in the riskless asset are allowed but that short positions in the riskless asset (borrowing) are not allowed. In both cases, we have assumed that an investor can take unlimited long or short positions in the risky assets.

In both cases, we find that the expected return on any risky asset is a linear function of its β, just as it is without any restrictions or borrowing. If there is a riskless asset, then the slope of the line relating the expected return on a risky asset to its β must be smaller than it is when there are no restrictions on borrowing. Thus a model in which borrowing is restricted is consistent with the empirical findings reported by Black, Jensen, and Scholes.

In both cases, the risky portion of every portfolio is a weighted combination of portfolios m and z, where portfolio m is the market portfolio, and portfolio z is the minimum-variance zero-β portfolio. Portfolio z has a covariance with risky asset i proportional to $1 - \beta_i$. If there is a riskless asset, then the efficient portfolios that contain the riskless asset are all weighted combinations of the riskless asset and a single risky portfolio t. Portfolio t is the efficient portfolio of risky assets with the highest ratio of the expected difference between the return on the portfolio and the return on the riskless asset to the standard deviation of the return on the portfolio. The line relating the expected return on an efficient portfolio to its β is composed of two straight line segments, where the segment for the lower-risk portfolios has a greater slope than the segment for the higher-risk portfolios.