AN EMPIRICAL EVALUATION OF ALTERNATIVE PORTFOLIO-SELECTION MODELS*

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I. INTRODUCTION

The approach currently regarded as providing the best analytical framework for selecting securities for an investment portfolio was first set forth by Markowitz and is described in detail in his 1959 book.[1] The Markowitz approach, however, has not as yet led to satisfactory solutions to the major problems of a real-world portfolio manager. One reason is that, like its predecessors, the Markowitz model greatly simplifies reality at a number of points, for example, by treating portfolio selection as a one-time act rather than as a continuous process of review and reallocation, subject to transactions and information costs. In addition, the full Markowitz model, with its more explicit recognition of the possible interrelationships among the performances of different securities, imposes new estimating demands on the security analyst and computational demands on the computer, demands which rise very rapidly as the number of securities to be considered rises toward any realistic level.

The purpose of the research reported here was empirically to evaluate the ex ante and ex post performances of a number of single-period portfolio-selection models based upon the Markowitz formulation but representing successive steps toward simpler (but less rigorous) models which pose fewer problems of data preparation and computation. These simpler models represent the covariance relationships between individual securities and one or more indexes of industry or market performance.

For the securities and period studied, our results indicate that the ex post performance of the index models is not dominated by the Markowitz formulation. It also indicates that, for strictly common stock universes, the performance of the multi-index models is not superior to that of the single-index formulation, indicating the secondary importance of industry considerations for common stock portfolios.

The ex post performance of the efficient sets was compared to that of randomly selected portfolios and actual performance of seventy-eight common stock mutual funds. The results indicate that, even with a naïve security evaluation model, the efficient sets dominate the random portfolios and are not dominated by the mutual funds.

II. FEATURES OF ALTERNATIVE PORTFOLIO-SELECTION MODELS

In order to apply the Markowitz technique, the investor must form expecta-

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[1] Harry M. Markowitz, *Portfolio Selection: Efficient Diversification of Investment* (Cowles Foundation Monograph No. 16 [New York: John Wiley & Sons, 1959]).

tions about the future performance of all securities in his universe. These expectations include not only the expected return and variance of return for each security but also the covariances between all possible pairs of returns. This requirement tends to be very large for security universes of practical size.

For example, in applying the Markowitz technique in a straightforward manner, 5,150 items of input data are required for an analysis of a 100-security universe. These data, in an operational situation, must be supplied almost entirely by the security analyst.

A large amount of computation time is required to handle an analysis of practical size. For example, an analysis of a 150-security universe using an existing computer program[2] required ninety minutes of IBM 7090 processing time. At the current commercial rate, the cost for this run is $600.

The index models incorporate extensions of the basic Markowitz framework, having been developed to simplify the data preparation problem and to allow the use of efficient computational algorithms which take advantage of the special properties of the index structure.

Algebraic expressions for the return and variance of a portfolio as a function of the component security data are developed in the Appendix for each model. In addition, the implicit correlation between individual securities in the universe implied by each of the models[3] is derived. The notation to be used throughout the paper will now be developed, followed by a brief discussion of each of the portfolio-selection models and the assumptions that each makes about the interactions among securities.

N = number of securities in the universe considered;
M = number of industry classifications into which the universe of N securities has been divided;
N_j = the *set* of securities in classification j, that is

$$\{i | i \epsilon N_j\}\ , \qquad j = 1, \ldots, M\ ;$$

R_i = a random variable, the distribution of which defines the possible returns on security i during some fixed time period where $i = 1, \ldots, N;$
$E(R_i)$ = expected value of $R_i;$
σ_{ii} = variance of the distribution of $R_i;$
σ_{ij} = covariance between returns R_i and $R_j;$
X_i = the proportion of the portfolio invested in security i where $X_i \geq 0$ for all i

$$\sum_{i=1}^{N} X_i = 1\ .$$

For compactness of notation, the following vector quantities are defined

X' = portfolio vector $= (X_i, \ldots, X_N);$
R' = return vector $= (R_i, \ldots, R_N);$
E' = expected value of return vector $= [E(R_i), \ldots, E(R_N)];$
$\mathbf{\Sigma}_N$ = covariance matrix for security returns $R_i, \ldots, R_N$
$= E\{[R - E(R)]\,[R' - E(R')]\}$
$= ||{}_{ij}\sigma||, i = 1., , , N, j = 1, \ldots, N;$
R_p = a random variable representing the return on a portfolio during a specified time period
$= R'X;$
E $= E(R_p)$ = expected value of $R_p = E'X;$
V $= VAR(R_p)$ = variance of R_p
$= X'\mathbf{\Sigma}_N X.$

[2] IBM Portfolio Selection Program (IB PS90), IBM 7090 Program No. 7090-FI 03X. Subsequent to completing most of the computations with this program, an improved Quadratic Programming Code was made available through the Share General Program Library. This program (RS-QPF4) considerably reduces the computation time required to generate an efficient set.

[3] Once the parameters for an index model have been obtained, the expressions for the implicit covariability between the returns of each pair of securities could be used to derive the necessary inputs for the Markowitz model. Identical results would then be achieved by both formulations. This approach was originally suggested by Markowitz (*op. cit.*, pp. 96–101) as a method of reducing the data input requirements for his formulation.

THE MARKOWITZ MODEL[4]

This model requires the following data for each component of the security universe considered.

1. The expected return, $E(R_i)$, $i = 1, \ldots, N$.
2. The variance of return, σ_{ii}, $i = 1, \ldots, N$.
3. The covariance of return between R_i and $R_{i'}$ for all pairs of securities, $\sigma_{ii'}$, $i \neq i'$, $i = 1, \ldots, N$, $i' = 1, \ldots, N$.

In effect, the investor must supply the expected return vector $\boldsymbol{E}$ and the covariance matrix $\boldsymbol{\Sigma}_N$.

The expected return and variance of any portfolio can be expressed in terms of the basic data [$E(R_i)$ and σ_{ij} values] and the amounts invested in various securities:

$$E = \sum_i X_i E(R_i) = \boldsymbol{X}'\boldsymbol{E},$$

$$V = \sum_i \sum_j X_i X_j \sigma_{ij} = \boldsymbol{X}'\boldsymbol{\Sigma}_N \boldsymbol{X}.$$

Given the above data, the model generates efficient portfolios which have minimum risk for any given level of return. Note that in this model no simplifying assumptions have been made regarding the relationships among securities. For an analysis of N securities, the analyst must provide estimates of N expected returns, N variances of return, and $N(N - 1)/2$ covariances of return.

SINGLE-INDEX MODEL

The practical application of the above technique would be greatly facilitated by a set of assumptions which would reduce the estimation task. One such set of assumptions is the *single-index model*. This approach, first suggested by Markowitz as a method of preparing input for the first model described, was later developed by Sharpe in a way that took computational advantage of the structure of the data.[5]

The major characteristic of the single-index model is the assumption that the various securities are related only through common relationships with an index of general market performance. In this model, the return from any security is determined by random factors and a linear relationship with the market index:

$$R_i = A_i + B_i I + C_i,$$

where A_i and B_i are parameters which can be determined for each security by least-squares regression analysis. C_i is a random element. I is the level of some index, such as the Dow-Jones Industrial Average, GNP, or any specially constructed index that is more closely aligned with the specific purposes of the analysis. The future level of the market index is given by

$$I = A_{N+1} + C_{N+1},$$

where

$$E(I) = A_{N+1},$$
$$E(C_{N+1}) = 0,$$
$$VAR(I) = E(C_{N+1}^2) = Q_{N+1}.$$

These relationships imply that the parameter A_{N+1} is an unbiased estimate of the market level in the time period considered. The possible values of I are thus distributed symmetrically about A_{N+1} with variance Q_{N+1}. In this formulation, the following assumptions are made

$$E(C_i) = 0, \quad i = 1, \ldots, N; \quad (1)$$

that is, for each security we have an unbiased estimate of the mean return

[4] For a more detailed description, see H. M. Markowitz, "Portfolio Selection," *Journal of Finance*, III, No. 1 (March, 1952), 77–91.

[5] William F. Sharpe, "A Simplified Model for Portfolio Analysis," *Management Science*, IX, No. 2 (January, 1963), 277–93.

during the time horizon given by

$$E(R_i) = A_i + B_iE(I) = A_i + B_iA_{N+1} .$$

$$VAR\,(C_i) = E(C_i^2) = Q_i\,, \qquad i = 1, \ldots, N\,, \qquad (2)$$

$$E(C_{N+1} \cdot C_i) = 0\,, \qquad i = 1, \ldots, N\,;$$

that is, for any given value of I, the return on security i is distributed about $A_i + B_iI$. Q_i, the variance of the residual return C_i, is independent of the level of I.

$$E(C_iC_{i'}) = 0\,, \qquad i \neq i'\,, \qquad i = 1, \ldots, N\,. \qquad i' = 1, \ldots, N\,; \qquad (3)$$

that is, the yield residuals are uncorrelated. This assumption states that the returns on any two securities i and i' are related only through the relationship with the market index I.

The contribution made by Sharpe was to show that, if you start out with the linear relationships defined above, then by appropriately introducing a new variable, a dummy $N + 1^{ST}$ security, a special type of covariance matrix is obtained. The covariance matrix, which is full in the N securities Markowitz formulation, contains non-zero elements only along the $N + 1$ diagonal positions for the single-index model. Everywhere off the main diagonal $\mathbf{\Sigma}_{N+1}$ has zero elements. This vastly reduces the amount of computation required to generate the efficient set of portfolios, as the process requires repeated inversion of the covariance matrix. When a computational algorithm is used which takes advantage of the diagonal form of the covariance matrix, the efficient set of portfolios can be obtained at approximately 1 per cent of the computation cost required for the full Markowitz formulation.

The dummy $N + 1^{ST}$ security can be considered as a weighted responsiveness of security returns to movements in the market level I, as

$$X_{N+1} = \sum_{i=1}^{N} X_iB_i\,.$$

As shown in the Appendix, the return and variance of a portfolio are given by

$$E(R_p) = \sum_{i=1}^{N+1} X_iA_i = \boldsymbol{X}'\boldsymbol{A}\,,$$

where

$$\boldsymbol{X}' = (X_i, \ldots, X_{N+1})\,,$$

and

$$\boldsymbol{A}' = (A_i, \ldots, A_{N+1})\,.$$

$$VAR(R_p) = \sum_{i=1}^{N+1} X_i^2Q_i = \boldsymbol{X}'\mathbf{\Sigma}_{N+1}\boldsymbol{X}\,.$$

While reduction in computation time is an important factor in the research and development phases of portfolio-selection models, the most important characteristic of this model with regard to operational application is the reduction in security data required. Only three estimates are required for each security, A_i, B_i, Q_i, and two for the market index, A_{N+1}, Q_{N+1}, rather than estimates of each element in the variance-covariance matrix. Thus the number of estimates required is reduced from $N(N + 3)/2$ in the Markowitz-model formulation to $3N + 2$ in the single-index formulation.

MULTI-INDEX MODELS

The single-index model presents a great simplification, in terms of both necessary inputs and computation, over the original Markowitz formulation. The question arises, however, as to whether this formulation is an oversimplification. In tying the variability of security yield only to a general market index, some of the important relationships among securities—originally expressed in the Markowitz formulation as independently determined covariances between each pair of securities—may be lost. It is hence possible that the original single-index model does not generate a truly efficient set of portfolios.

This potential deficiency would be particularly acute when several classes of securities are being considered for inclusion in the same portfolio. It would not appear realistic that a single index of market performance would provide an adequate base for expectation about the future performance of common stocks, preferred shares, bonds, and other types of assets.

Part of the purpose of this research is to investigate empirically the differences in the results from the Markowitz and single-index Sharpe formulations for universes of common stocks. Given the wide gap between the relatively rigorous method by which the traditional Markowitz technique treats the relationships between securities and the very simplified way this is done in the single-index model, it seemed appropriate to consider other models of intermediate complexity between these two extremes. This would allow us to capture the covariance relationships in a potentially more efficient manner than the single-index model, while at the same time achieving some computational savings over the general Markowitz approach.

To this end, we have developed two multi-index models; one we call the *covariance form* of the *multi-index model;* the other, the *diagonal form* of the *multi-index model.*

All of the index models are similar in that they relate the return for any security as a linear function of some index. However, in the multi-index model, rather than using a general market index we use a number of class or industry indexes. In the case of strictly common stock universes, the class indexes can be thought of as industry indexes. In dealing with different classes of securities, such as preferred stocks, common stocks, government bonds, corporate bonds, and so on, a special appropriate index could be defined for each of these classes of securities.

The first multi-index model, the covariance form, maintains the single-index type of formulation within each security class. It allows, however, for the full covariability among class indexes, in much the same manner as between securities in the Markowitz formulation.

The second model, the diagonal form, employs a hierarchy of indexes. The first group of indexes relates directly to the yields of the securities in their respective classes of industry groupings, in the same manner as the covariance model. An additional index is then used as a medium for expressing the relationships among the various industry or class indexes. The structures and assumptions of these models will now be discussed in more detail.

Multi-index model—covariance form.— In this model we assume that the universe of securities is composed of components from M classes or industries. The return on each security is assumed to be linearly related to the level of the index of the industry or class to which it belongs.

$$R_i = A_i + B_i J_j + C_i\,, \quad \{i \mid i \epsilon N_j\}\,,$$

where $N_j =$ the *set* of securities in class j, $j = 1, \ldots, M$. J_j is the future level of jth industry index, where $J_j = A_{N+j} + C_{N+j}$, $j = 1, \ldots, M$. As in the single-index model, the following assumptions are made

$$E(C_i) = 0\,, \qquad i = 1, \ldots, N\,. \quad (1)$$

$$E(C_i \cdot C_{i'}) = Q_i\,, \quad i = i'\,, \quad i = 1, \ldots, N\,,$$
$$= 0\,, \quad i \neq i'\,, \quad i = 1, \ldots, N\,. \tag{2}$$

$$E(C_{N+j}) = 0\,, \quad j = 1, \ldots, M\,. \tag{3}$$

$$E(C_{N+j}^2) = Q_{N+j}\,, \quad j = 1, \ldots, M\,. \tag{4}$$

$$E(C_{N+j}\, C_i) = 0\,, \quad i = 1, \ldots, N\,, \quad j = 1, \ldots, M\,. \tag{5}$$

Thus far the assumptions *within* each class are similar to those in the single-index model. To express the relationships among the M industry subuniverses, we introduce the covariance matrix of the industry indexes

$$\boldsymbol{\Sigma}_M = ||\sigma_{jj'}|| = ||COV\, J_j \cdot J_{j'}||\,,$$
$$j = 1, \ldots, M\,. \quad j' = 1, \ldots, M\,.$$

As shown in the Appendix, the expressions for the expected return and variance of a portfolio are given by

$$E(R_p) = \sum_{i=1}^{N+M} X_i A_i = \boldsymbol{X}'\boldsymbol{A}\,,$$

where

$$X_{N+j} = \sum_{\{i | i \epsilon N_j\}} X_i B_i\,, \qquad j = 1, \ldots, M\,,$$
$$\boldsymbol{X}' = (X_1, \ldots, X_{N+M})\,,$$
$$\boldsymbol{A}' = (A_1, \ldots, A_{N+M})\,.$$

$$VAR\,(R_p) = \sum_{i=1}^{N} X_i^2 Q_i + \sum_{j=1}^{M} \sum_{j'=1}^{M} X_{N+j} X_{N+j'} \sigma_{jj'}$$
$$= \boldsymbol{X}_N' \boldsymbol{\Sigma}_N \boldsymbol{X}_N + \boldsymbol{X}_M' \boldsymbol{\Sigma}_M \boldsymbol{X}_M$$
$$= \boldsymbol{X}' \boldsymbol{\Sigma}_{N+M} \boldsymbol{X}\,,$$

where

$$\boldsymbol{\Sigma}_{N+M} = \begin{vmatrix} \boldsymbol{\Sigma}_N & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Sigma}_M \end{vmatrix}\,.$$

The covariance matrix ($\boldsymbol{\Sigma}_{N+M}$), which must be repeatedly inverted in generating the efficient frontiers, can be partitioned into four submatrixes, only two of which have non-zero elements.

The first submatrix ($\boldsymbol{\Sigma}_N$) is a diagonal matrix because of the single-index assumptions within each industry. The second matrix ($\boldsymbol{\Sigma}_M$) is not simplified at all because we have made *no* simplifying assumptions regarding the covariances among industry indexes. However, a great deal of computational saving will be realized in a realistic application, since the number of industry indexes (M) will be much fewer than the number of securities (N). Thus when the total covariance matrix $\boldsymbol{\Sigma}_{N+M}$ is inverted using partitioning techniques, the only matrix which must be inverted using general techniques is the smaller $\boldsymbol{\Sigma}_M$ matrix. The inverse of $\boldsymbol{\Sigma}_N$, being a diagonal matrix, is easily and quickly obtained.

Multi-index model—diagonal form.—This model has the same basic structure as the covariance form, with the additional assumption that each industry index is itself linearly related to an over-all market index. This involves the definition of a further dummy security (the $N + M + 1^{ST}$) which is related to the responsiveness of the industry indexes to the general market index.

The future levels of the industry indexes are thus assumed to be given by: $J_j = A_{N+j} + B_{N+j}\, I + C_{N+j}, j = 1, \ldots, M$, where we make similar assumptions to those made in the single-index formulation or within an industry group as in the covariance model:

$$E(C_{N+j}) = 0\,, \quad j = 1, \ldots, M\,.$$
$$E(C_{N+j}^2) = Q_{N+j}\,, \quad j = 1, \ldots, M\,.$$
$$E(C_{N+j} C_i) = 0\,, \quad i = 1, \ldots, N\,, \quad j = 1, \ldots, M\,.$$
$$E(C_{N+j} \cdot C_{N+j'}) = 0\,, \quad j \neq j'\,.$$

The level of the general market index, I, is defined as in the single-index model $I = A_{N+M+1} + C_{N+M+1}$, where A_{N+M+1} is the expected value of I and C_{N+M+1} is a random variable with mean zero and variance Q_{N+M+1}. C_{N+M+1} is assumed to be uncorrelated with any of the other security or index residuals, that is, $E(C_{N+M+1} \cdot C_i) = 0, i = 1, \ldots, N + M$. As developed in the Appendix, the expressions for the return and variance of a portfolio are given by

$$E(R_p) = \sum_{i=1}^{N+M+1} X_i A_i = \boldsymbol{X}'\boldsymbol{A},$$

where

$$X_{N+j} = \sum_{\{i | i \epsilon N_j\}} X_i B_i,$$

$$X_{N+M+1} = \sum_{j=1}^{M} X_{N+j} B_{N+j},$$

$$\boldsymbol{X}' = (X_1, \ldots, X_{N+M+1}),$$

$$\boldsymbol{A}' = (A_1, \ldots, A_{N+M+1}).$$

$$VAR(R_p) = \sum_{i=1}^{N} X_i^2 Q_i + \sum_{j=1}^{M} X_{N+j}^2 Q_{N+j} + X_{N+M+1}^2 Q_{N+M+1}$$

$$= \sum_{i=1}^{N+M+1} X_i^2 Q_i$$

$$= \boldsymbol{X}'\boldsymbol{\Sigma}_{M+N+1}\boldsymbol{X},$$

where $\boldsymbol{X}' = (X_1, \ldots, X_N, X_{N+1}, \ldots, X_{N+M+1})$.

When the form of the covariance matrix $\boldsymbol{\Sigma}_{N+M+1}$ is examined, it is found to be completely diagonal, as it was in the case of the single-index model. It is not the same covariance matrix, however, because even though we are in a sense relating each security ultimately to a market index, due to the differences in the assumptions about the properties of the yield and index residuals, the covariance matrixes will be different.[6]

COMPARISONS OF THE MODEL FORMULATIONS

To summarize at this point, it is seen that we will be considering four different versions of the efficient frontier. The four models theoretically form a decreasing sequence with respect to the completeness by which each model represents the true covariability between the securities of the universe. Starting with the complete Markowitz formulation, we have an exact representation of the covariance relationships.

Next we have the multi-index model, covariance form, where the universe has been divided into classes or industries. The relationships among the industry indicators are completely maintained in this model by the inclusion of a full variance-covariance matrix for these indexes.

In the next model, the diagonal form of the multi-index model, we attempt to relate the levels of the industry indexes through their relationship with a common index of market level. Thus instead of an exact representation of the covariability of the indexes, we are now using a linear model, with its inherent assumptions about homogeneity of variance and non-correlation of residuals. Hence this is a less complete representation than the preceding model.

In the final model, the single-index formulation, we have made the assumption that the returns on all securities in the universe are related only through their common dependence on the general market index. However, along with this decreasing ability to represent the true covariance matrix, comes increasing ease of computation or decreasing of compu-

[6] This can be seen by comparing the algebraic expressions for the implicit correlation between pairs of returns for both models (refer to the Appendix).

tation costs because each of these models generates a covariance matrix that is successively easier to invert.

Rather than developing specific computationally efficient programs for each of the index models, we have used an existing general-purpose portfolio-selection computer program[7] to in effect simulate the structure of each of our models. The program, being very general in nature, does not make use of computational efficiencies which are inherent in the data structure of the various index models. Thus we cannot make statements about the computational properties of the index models other than to specify a computational ranking based on our knowledge of the structures of the models. We are more concerned at this time with empirically investigating the relative performance of these models. Given the superiority of one formulation in a particular circumstance, it would then be appropriate to develop an efficient, computational code, if one does not already exist.[8]

III. DATA AND TESTS USED

The test samples of 75- and 150-security universes have been prepared using yearly price and yield data for the periods 1947–57 (ex ante) and 1958–64 (ex post).

The ex ante efficient portfolios generated by each of the four models have been examined to compare (1) the location of the ex ante efficient frontiers; (2) the composition of the efficient portfolios; (3) the performance of the ex ante efficient portfolios during the ex post period.

In addition, the ex post performance of the efficient sets has been compared to that of randomly generated portfolios and the actual performance of seventy-eight basically common stock mutual funds.

In order that the results of the research be more meaningful to the institutional investor, we have placed upper-bound constraints on the amount of any security which can be contained in an efficient portfolio. In practice, many institutional investors have legal restrictions on the proportion of their portfolio which can be invested in any one security. Others adhere heuristically to such restrictions to avoid becoming formally involved as major shareholders in the companies in which they invest. In other cases, it may still be desirable to employ upper-bound restrictions as a method of hedging against the risk of biases in the input data.

Formally, these limits can be considered to be upper-bound constraints on the variables X_i. Such upper bounds have been introduced into all four of the portfolio-selection models with which we deal in this paper. When efficient portfolios have been generated from a universe of seventy-five common stocks, the upper-bound constraints have all been set equal to 0.05, insuring that a minimum of twenty securities appears in each portfolio; when a universe of 150 stocks has been used, all the upper bounds have been equated to 0.025, so that the minimum number of securities in a portfolio is forty.

Before presenting a description of the empirical findings, some of the considerations involved in developing input data for the study will now be discussed.

DEVELOPMENT OF INPUT DATA

Yearly security data for the period 1947–64 were used to develop input for

[7] IBM Portfolio Selection Program, *op. cit.*

[8] IBM has developed a 1401 computer code for the single-index model: 1401 Portfolio Selection Program (1401-FI-04 X).

the portfolio selection and evaluation phases of the study. The source of this information was the Standard and Poor's Compustat Industrial Service. Although this included over nine hundred common stocks, only 543 had the necessary continuous price and dividend histories over the full 1947–64 period. The data were arbitrarily divided into two groups, 1947–57 and 1958–64. Security information from the initial eleven-year period was used to develop the required estimates for each portfolio-selection model. Data from the final seven-year period were used to evaluate the ex post performances of the sets of efficient portfolios.

To measure the yield for a security in any year, both capital gains and dividends were considered.[9] For simplicity, tax effects were not considered. Yields were computed for each of the 543 securities in each year of the 1947–64 period. These yields were then used to develop market and industry indexes for the index models.

Rather than using any of the standard published indexes, an aggregate performance index was computed which was more pertinent to the investment performance of our security universe. This index used is an unweighted arithmetic average of yields of all securities in the 543-security universe.[10] The universe of securities was divided into ten industry subgroupings, and similar industry indexes were computed for each industry.

In order to generate the expected values of returns for the Markowitz model and the expected value of the indexes for the index models, the following assumptions were made:

1. The expected return for each security for the period 1958–64 was assumed to be an arithmetic average[11] of the yearly returns in the initial period.

2. Similarly, the expected value (A_{N+1}) for each industry or general index was assumed to be an average of the actual levels in the 1947–57 period.

3. Similar assumptions were made regarding variability and covariability of security yield and index level. Estimates of future variability were assumed to be equal to those computed for the initial period.

4. The expected future values of the parameters for the index models (A_i, B_i, Q_i) were assumed to be equal to the values developed in the 1947–57 period using least-squares regression techniques.

In effect, we are assuming that performance in our seven-year evaluation period can be adequately predicted from

[9] Yields were calculated for each year in the following manner: $R_i(t) = [\text{Price}_i(t) + \text{Dividends}(t) - \text{Price}_i(t-1)]/[\text{Price}_i(t-1)]$, where $R_i(t)$ is the yield on security i in year t.

[10] For the rationale underlying this type of index, see Kalman J. Cohen and Bruce P. Fitch, "The Average Investment Performance Index," *Management Science*, Series B, XII, No. 6 (February, 1966), B-195–B-215.

[11] We use the arithmetic rather than geometric average here because we are not interested in the average compounded rate of growth of a portfolio over a successive number of years. Rather, since we are dealing with static selection models, in which the definition of time horizons is arbitrary, we prefer to consider our ex post and ex ante periods as effective "single-year" periods in which the return vectors $R = (R_1, \ldots, R_N)$ are independently distributed according to the probability distributions $f_i(R_1, \ldots, R_N)$, $i = 1$ (ex ante period), 2 (ex post period).

Thus the eleven observations arbitrarily allocated to the ex ante "period" can be assumed to be random and independent observations from the population of "one-year" returns. As such we use a least-squares, or arithmetic averaging, technique to estimate the mean of the ex ante distribution $f_i(R_1, \ldots, R_N)$.

As of the end of 1957, when the portfolios are selected, the moments of $f_1(R_1, \ldots, R_N)$ are assumed to be the best estimates of the unknown moments of $f_2(R_1, \ldots, R_N)$. In developing the actual moments of $f_2(R_1, \ldots, R_N)$ for evaluating the ex post performance of the ex ante efficient portfolios, a similar argument applies, i.e., the seven years of data can be assumed to be seven random and independent observations from $f_2(R_1, \ldots, R_N)$.

the performance during the eleven-year base period. In order to avoid possible misunderstandings, we must stress that in an operational situation we would definitely *not* advocate any method of forming future expectations which is based strictly on historical data. We have adopted such a method in this study because we are concerned at this time with only a part of the portfolio-analysis process. The naïveté of our security-evaluation model should not change any conclusions we may wish to make about the *relative* performance of various portfolio-selection techniques.

When the efficient frontiers had been generated, the yield data from the 1958–64 period were used to calculate the *true* ex post return and variance of the efficient portfolios. The computation method for all models was that specified for the Markowitz formulation, using the true covariance matrix.

All calculations were carried out for both the 150- and 75-security universes. The 150-security universe is a randomly chosen subset of the 543 common stocks available. The 75-security universe is a randomly chosen subset of the 150-security universe. This nesting of the universes was established so that the comparisons of the results obtained from the 75-security and the 150-security universes would primarily portray the effects of universe size rather than of differences in the nature of the securities.

To provide a basis for an objective comparison of the ex post performance of the efficient sets, the actual performances of randomly generated portfolios and some common-stock mutual funds were considered.

Two groups of sixty random portfolios were chosen, one group to correspond to each universe size, that is, such that the random portfolios would have approximately the same numbers of securities per portfolio as efficient portfolios selected from the respective universes.[12]

The seventy-eight mutual funds were selected from Table 19 of Arthur Wiesenberger's *Investment Companies*.[13] Those selected include all growth, growth and income, and income with growth funds which have continuous histories for 1958–64. The basic yearly "return" for the mutual funds used is defined as the percentage change in net-asset value per share plus capital gains distributions plus income dividends. The average return and variance of return for each mutual fund over the seven-year ex post period was computed in a straightforward fashion consistent with previous ex post calculations. In effect, each mutual fund was treated as a separate portfolio for evaluation purposes.

ANALYSIS OF THE CORRELATION ASSUMPTIONS IN THE INDEX MODELS

If the returns of all securities in the universe were related in such a manner that the various yield *and* index residuals in each model were absolutely uncorrelated, that is, $E(C_i \cdot C_j) = 0$, for $i \neq j$, over the time horizon considered, then the index models would represent the true covariability of the securities identically. However, the assumption that the residuals are uncorrelated is an approximation. By assuming the various residuals to be uncorrelated for the purpose of model formulation, the *implied* covariances among securities in the index models will differ from the true covari-

[12] For comparison with the 75-security universe efficient portfolios, the random portfolios were selected to contain 20 securities. For the 150-security universe, each random portfolio consisted of 40 randomly selected securities. Equal dollar weights were given to the securities contained in each random portfolio.

[13] A. Wiesenberger, *Investment Companies* (Port Washington, N.Y.: Kennikat Press, 1965).

ance as defined by the Markowitz model.

Table 1 shows the distributions of correlation coefficients among the *yield* residuals (C_i, $i = 1, \ldots, 150$) for the single-index and multi-index models for the 1947–57 period. These distributions, while centered about zero, have reasonably wide dispersion. By assuming away this yield correlation, we are in effect reducing the covariability between securities, which in turn will cause the "reduced" covariance matrixes implied by these models to understate the variance of efficient portfolios generated by them.

TABLE 1

DISTRIBUTIONS OF CORRELATION COEFFICIENTS OF YIELD RESIDUALS, 150-SECURITY UNIVERSE, 1947–57

Correlation Coefficient	Single-Index Model		Multi-Index Model (Covariance Form)	
	Relative Frequency	Cumulative Frequency	Relative Frequency	Cumulative Frequency
−1.000 to − .900	.001	.001	.000	.000
− .899 to − .800	.004	.005	.002	.002
− .799 to − .700	.013	.018	.009	.011
− .699 to − .600	.027	.045	.023	.034
− .599 to − .500	.041	.086	.037	.071
− .499 to − .400	.058	.144	.058	.129
− .399 to − .300	.077	.221	.076	.205
− .299 to − .200	.089	.310	.090	.295
− .199 to − .100	.089	.399	.098	.393
− .099 to .000	.101	.500	.104	.497
.000 to .099	.101	.601	.105	.602
.100 to .199	.093	.694	.104	.706
.200 to .299	.086	.780	.086	.792
.300 to .399	.073	.853	.076	.868
.400 to .499	.058	.911	.054	.922
.500 to .599	.042	.953	.039	.961
.600 to .699	.028	.981	.024	.985
.700 to .799	.015	.996	.012	.997
.800 to .899	.004	1.000	.003	1.000
.900 to 1.000	.000	1.000	.000	1.000

It is interesting to note that the distribution of correlation coefficients for the multi-index model is only slightly less dispersed than that for the single-index model. Thus the structuring of the models to include a number of indexes has not had as major an effect on reducing the covariability among yield residuals for the universe of common stocks considered as might have been expected.

TABLE 2

DISTRIBUTION OF CORRELATION COEFFICIENTS BETWEEN INDEXES, MULTI-INDEX MODEL, COVARIANCE FORM, 1947–57

Correlation Coefficient	Relative Frequency	Cumulative Frequency
.000 to .099	.000	.000
.100 to .199	.000	.000
.200 to .299	.000	.000
.300 to .399	.000	.000
.400 to .499	.000	.000
.500 to .599	.045	.045
.600 to .699	.045	.090
.700 to .799	.244	.334
.800 to .899	.333	.667
.900 to 1.000	.333	1.000

Table 2 summarizes the distribution of correlation coefficients among the ten-industry indexes for the period 1947–57. The very high interrelations among industries is very evident from this table. This high interindex correlation would not be as predominant if we were dealing with a wider class of securities than just common stocks.

The structure of the covariance form of the multi-index model includes a 10×10 covariance matrix to account exactly for the correlations among industry indexes. In the diagonal form, each industry index is assumed to be linearly related to the general market index. As in the single index model, the index residuals (C_{N+j}, $j = 1, \ldots, M$) are assumed to be mutually uncorrelated. Table 3 indicates the empirical results of this assumption. While a large amount of the covariability between industry indexes is explained by their common dependence on the general market index, the assumption does not fit the facts as well as in the yield residuals case. The dispersion of the correlation coefficient distribution is wider and somewhat skewed.

While Tables 1, 2, and 3 are interesting insofar as they indicate how well some of our individual assumptions are satisfied, a more aggregate measure, which is perhaps more meaningful to the final selection performance of the models, is how well the models are able to reproduce the true covariances between individual security returns. To obtain a measure of this relative ability, the correlation matrix implied by each of the index models was compared with the true correlation matrix used in the Markowitz model.[14] While the multi-index models most closely represented the true correlations among securities within the same industries, the relationships among securities in different industries were somewhat better represented by the single-index model. Because of the much larger number of interindustry as opposed to intraindustry comparisons, the single-index model was found, on the average, to better represent the true correlation matrix. Table 4 indicates the distributions of coefficient differences for the single and multi-index (covariance form) mod-

TABLE 3

DISTRIBUTION OF CORRELATION COEFFICIENTS OF INDEX RESIDUALS, MULTI-INDEX MODEL, DIAGONAL FORM, 1947–57

Correlation Coefficient	Relative Frequency	Cumulative Frequency
−1.000 to − .900	.000	.000
− .899 to − .800	.089	.089
− .799 to − .700	.000	.089
− .699 to − .600	.067	.156
− .599 to − .500	.089	.245
− .499 to − .400	.111	.356
− .399 to − .300	.022	.378
− .299 to − .200	.067	.445
− .199 to − .100	.111	.556
− .099 to .000	.044	.600
.000 to .099	.022	.622
.100 to .199	.089	.711
.200 to .299	.067	.778
.300 to .399	.089	.867
.400 to .499	.000	.867
.500 to .599	.067	.934
.600 to .699	.022	.956
.700 to .799	.022	.978
.800 to .899	.022	1.000
.900 to 1.000	.000	1.000

[14] The true correlation matrix, as used in the Markowitz formulation, was compared with the implicit correlation matrix of each of the index models for the 75-security universe. Each matrix consisted of 2,775 coefficients above the main diagonal, of which 2,484 had been generated by interindustry security correlations, and only 291 coefficients were the result of correlations among securities in the same industries.

Differences were taken between the true correlation coefficients and the equivalent correlation coefficients generated by each index model. The distributions of intraindustry and interindustry differences were then compared.

The distribution of differences for the 291 intraindustry correlations was more tightly distributed about zero for the multi-index model. However, the distribution of the 2,484 interindustry differences was slightly more centralized for the single-index model. Thus, when the differences for the total matrix were examined, the single-index model was found, on the average, to better represent the true correlation matrix (see Table 4).

els for the 75-security universe for the period 1947–1957.

The reason for this slight superiority is that the compound assumptions required to introduce the multi-index structure appear to introduce more error into the implicit correlation between two securities in different industries than the single-index model. This is felt to be the result of dealing with strictly common stock universes, in which the industries tend to be strongly interrelated and amenable to the single-index type of assumptions. If a wider class of securities had been included, it is felt that the multi-index formulation would tend to dominate in interindustry as well as intraindustry comparisons.

When the implied correlation matrixes of the covariance and diagonal forms of the multi-index model are compared, the former is found to dominate slightly the latter in its ability to reproduce the true correlations, as would be expected.

TABLE 4

DISTRIBUTION OF CORRELATION COEFFICIENT ERROR, 75-SECURITY UNIVERSE, 1947–57

Correlation Coefficient Error	Single-Index Model		Multi-Index Model (Covariance Form)	
	Relative Frequency	Cumulative Frequency	Relative Frequency	Cumulative Frequency
−1.000 to − .900	.000	.000	.000	.000
− .899 to − .800	.000	.000	.000	.000
− .799 to − .700	.000	.000	.000	.000
− .699 to − .600	.001	.001	.003	.003
− .599 to − .500	.003	.004	.006	.009
− .499 to − .400	.013	.017	.020	.029
− .399 to − .300	.034	.051	.042	.071
− .299 to − .200	.074	.125	.080	.151
− .199 to − .100	.124	.249	.117	.268
− .099 to .000	.204	.453	.176	.444
.000 to .099	.221	.674	.186	.630
.100 to .199	.156	.830	.163	.793
.200 to .299	.092	.922	.103	.896
.300 to .399	.044	.966	.064	.960
.400 to .499	.020	.986	.024	.984
.500 to .599	.011	.997	.014	.998
.600 to .699	.003	1.000	.002	1.000
.700 to .799	.000	1.000	.000	1.000
.800 to .899	.000	1.000	.000	1.000
.900 to 1.000	.000	1.000	.000	1.000

IV. RESULTS AND ANALYSIS

Figures 1 and 2 illustrate the ex ante efficient frontiers for the two security universes considered.

Figure 1 shows the efficient frontiers as specified by the selection models. In the index-model cases, the standard deviation levels associated with the frontiers are understated to varying degrees, having been computed by the "reduced" covariance matrixes implicit in these models. The amounts by which the risk levels have been understated can be seen by comparing Figure 1 with Figure 2. In Figure 2 the actual ex ante standard deviations associated with the various efficient portfolios have been calculated using the true ex ante covariance matrix (as developed for the Markowitz model).

The results in Figure 2 allow direct comparisons between models, showing the relative optimization ability of each of the models for the security universes considered. While the performance of the covariance form of the multi-index model was found to dominate that of the diagonal model, the dominance was so slight that the curves have been combined.

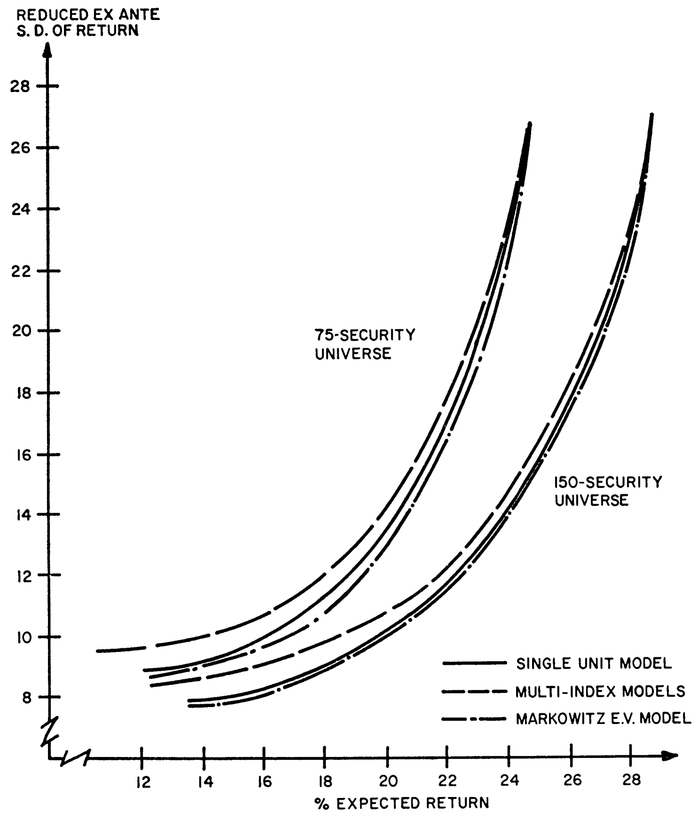

FIG. 1.—Comparison of efficient frontiers

The interesting feature of Figures 1 and 2 is the relative locations of the single- and multi-index frontiers. The single-index frontier tends to dominate those of the multi-index models over a wide range of expected returns. This is related to the ability of the single-index model to represent large parts of the true covariance matrix more effectively than do the multi-index models, for universes of strictly common stocks. As previously discussed, if we were dealing with a wider class of securities, one would expect this result to be reversed.

It is also noted that the single-index model tends to understate the standard deviations of efficient portfolios to a greater extent than the multi-index models. This is seen by comparing the relative upward shifts in proceeding from Figures 1 to 2. As seen from Figure 1, the dominance of the single-index model over the multi-index formulations is not nearly so clearly defined as in Figure 2.

These figures also illustrate the combined effects of jointly changing the universe size and the upper-bound restrictions. Increasing the universe size from 75 to 150 securities gives the models greater opportunity to increase return for ered, doubling the universe size had a much stronger effect than halving the upper-bound restrictions. In this case, it would be necessary to more than halve the upper-bound constraints to balance doubling the universe size. On the basis

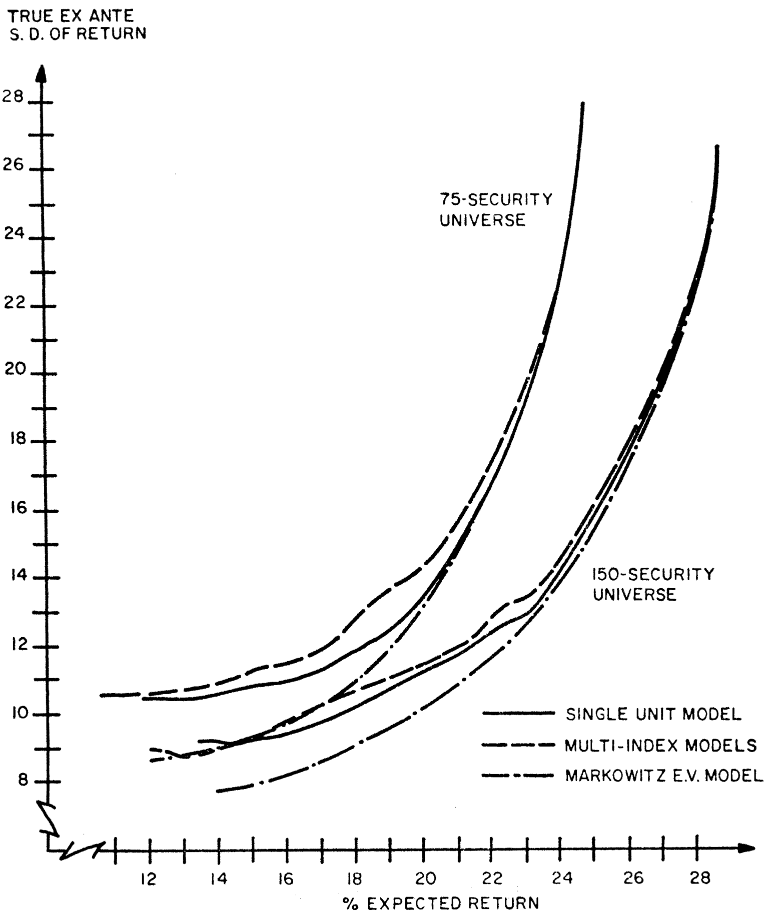

FIG. 2.—Comparison of efficient frontiers

the same level of risk. Hence we would expect that increasing the universe size should shift the efficient frontiers to the right. Conversely, decreasing the upper-bound restrictions for some given universe size should have the opposite effect. The results shown in these figures illustrate that, for the security sample considered, doubling the universe size had a much stronger effect than halving the upper-bound restrictions. In this case, it would be necessary to more than halve the upper-bound constraints to balance doubling the universe size. On the basis of this limited sample, we might conjecture that the ex ante expectations may be more sensitive to decisions which restrict the universe size than to decisions regarding the size of the upper-bound restrictions to be placed on the proportion of the portfolio that can be invested in any one security.

COMPOSITION OF THE EFFICIENT PORTFOLIOS

The efficient portfolios generated by each index model were compared to the Markowitz portfolios for equivalent levels of ex ante expected return. The comparisons between the non-optimal index portfolios and the optimal Markowitz portfolios consisted of computing the square root of the average sum of squared deviations between the investment proportions of the index-model portfolios and the investment proportions held by the optimal Markowitz portfolios; that is, the goodness of fit was measured by

$$D_{iE}=\sqrt{\frac{1}{N}\sum_{j=1}^{N}(X_{ij}-X_{4j})^2}$$

where

i = model number, $i = 1$ single index model,
$i = 2$ diagonal model,
$i = 3$ covariance model,
$i = 4$ Markowitz model;

j = security number, $j = 1, \ldots, N$ for $N = 75, 150$;

E = level of expected return at which the portfolios are being compared;

X_{ij} = proportion of the portfolio with E expected return invested in jth security by model i;

D_{iE} = a goodness-of-fit measure between two multivariate portfolios. It is analogous to the standard deviation of the portfolio composition of the ith model (at level E) when the optimal Markowitz portfolio ($i = 4$) is used as a standard of comparison.

The goodness-of-fit indexes have been plotted in Figure 2 for both universe sizes. The superior ex ante performance of the single index relative to the multi-index model is explained by observing how much more closely the single index portfolios "fit" the Markowitz portfolios over the whole range of returns, for both universe sizes.

TABLE 5

SUMMARY OF COMPOSITION STATISTICS

	75 Securities				150 Securities			
	Single	Diagonal	Covariance	Expected Return Variance	Single	Diagonal	Covariance	Expected Return Variance
Average number of securities in portfolio	24.3	23.7	23.5	22.6	44.7	44.4	44.3	42.9
Average number of securities at upper bound	16.0	16.6	16.9	18.3	35.2	35.9	35.9	37.6
Average percentage at upper bound	65.9	70.5	71.8	81.1	78.8	80.9	81.0	87.7

At the highest possible level of return, in both universes, the portfolios of the four models are identical, each consisting of the minimum number of securities allowed by the constraints having the highest expected returns. As the level of return decreases, the portfolios tend to become less similar as more opportunities for substitution at the same level of portfolio return become available.

Table 5 summarizes the average composition of the efficient portfolios across the range of expected returns. Note that, on the average, the optimal Markowitz portfolios tend to contain the fewest securities of which the highest percentage are at their upper bounds. As we proceed through the multi-index to the single-index models, the portfolios tend to contain more securities, of which a progressively higher percentage are at fractional or unconstrained levels. Thus the multi-index portfolios tend to be structurally more similar to the optimal portfolios, but as seen from Figure 3, they have a greater compositional variance.

In Tables 6 and 7 the composition of the ex ante portfolios is summarized at specific levels of expected return, showing the number of securities and the percentage of the portfolio invested in each industry grouping. These tables indicate how the composition of the efficient portfolios vary over the attainable range of expected returns.

COMPARISON OF EX POST PERFORMANCE

Figures 4 and 5 show the ex post results for a sample of efficient portfolios, plotted on the same diagram as the ex ante expectations, for the diagonal form of the multi-index model. In addition, the ex post performance of the randomly selected portfolios and seventy-eight mutual funds have been included for com-

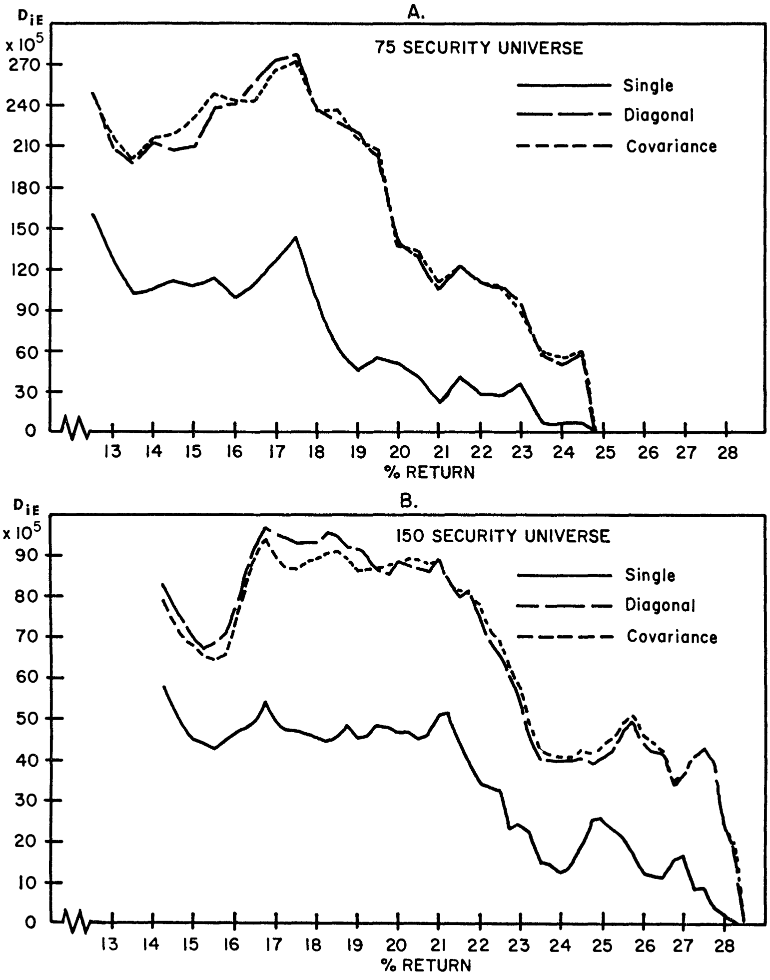

FIG. 3.—Goodness-of-fit test—efficient portfolios, Markowitz versus index models

TABLE 6

COMPARISON OF EFFICIENT PORTFOLIOS BY INDUSTRY GROUPINGS, 75-SECURITY UNIVERSE

A. NUMBER OF SECURITIES IN EACH INDUSTRY

INDUSTRY	No.	12.5%				14.0%				16.0%				18.0%				20.0%				22.0%				24.0%			
		1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Mines	5	1	1	1	0	0	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2
Foods	12	7	7	6	4	7	6	6	4	6	5	5	4	4	5	5	5	4	4	4	3	4	4	4	4	3	3	3	3
Textiles	2	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	0	1	1	1	1	1	1	0	0	1
Paper	5	1	1	1	1	1	1	1	1	0	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	2	2	2	2
Chemicals	8	4	3	2	3	4	2	2	2	4	2	2	3	2	2	2	3	2	2	2	2	1	2	2	2	1	1	1	1
ORSGC*	8	2	3	3	2	2	3	3	3	3	3	4	2	4	4	4	3	4	5	5	4	2	3	3	2	2	2	2	2
Metals	7	0	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1	2	2	1	1	1	1	1
Electricals	13	5	4	6	5	5	6	6	5	5	6	6	6	5	5	6	6	5	4	4	6	6	6	6	6	5	6	6	5
Transport	7	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	1	2	2	2	2
Retail	8	6	5	5	5	6	6	5	5	6	5	5	5	5	4	4	4	5	2	2	2	3	2	2	3	2	2	2	2
No. in portfolio	.....	27	25	25	22	26	26	25	24	27	24	25	23	24	22	24	24	24	22	21	21	22	23	23	22	21	21	21	21
No. at upper bound	.....	13	16	17	18	14	13	15	18	15	16	16	17	16	17	16	19	19	17	17	19	17	17	18	19	19	18	17	19
% at upper bound	.....	48.2	64.0	68.0	81.8	53.9	50.0	60.0	75.0	55.6	66.7	64.0	73.9	66.7	77.3	66.7	79.2	79.2	77.3	81.0	90.5	77.3	73.9	78.3	86.4	90.5	85.8	81.0	90.5

B. PERCENTAGE OF PORTFOLIO INVESTED IN EACH INDUSTRY

INDUSTRY	12.5%				14.0%				16.0%				18.0%				20.0%				22.0%				24.0%			
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Mines	3 2	5.0	5.0	0	0	4.8	5.0	1.9	1.7	2.2	2.8	4.7	3.6	0	1.2	5.0	5.0	3.4	3.2	5.0	5.0	5.0	5.0	5.0	6.6	7.4	7.5	6.7
Foods	25.8	26.1	24.0	16.2	22.3	22.8	21.4	16.1	22.9	22 8	22.5	20.0	16.4	22.0	21.7	16.6	15.4	20.0	20.0	15.0	19.6	15.2	15.2	18.1	15.0	15.0	15.0	13.3
Textiles	2 7	0	0	5.0	3.2	0	0	5.0	4.7	0	0	5.0	5.0	0	0	5.0	5.0	0.1	0	5.0	5.0	1.2	1.2	5.0	5.0	0	0	5.0
Paper	5 0	5.0	5.0	5.0	5.0	4.6	4.6	5.0	0	1.2	0.4	0	4.2	4 9	3.5	0	5.0	5.0	5.0	5.0	5.0	5 0	5.0	5.0	10.0	10.0	10.0	10.0
Chemicals	12 6	6.6	5.8	12.1	11.8	7.6	7.2	7.7	11.3	8 8	8.4	7.8	10.0	10.0	9.7	11.6	10.0	10.0	10.0	10.0	5.0	10 0	10.0	5.7	3 4	3.9	3.6	5.0
ORSGC*	10 0	11.2	12.1	10.0	10.0	12.3	13.4	12.2	11 2	13.7	15.2	10.0	15.3	18.3	19.9	15.0	20.0	22.6	22 6	18.1	10 0	11.3	11.2	10.0	10.0	10.0	10.0	10.0
Metals	0	4.4	5 0	0	0	4 8	5.0	1.7	0.4	5.0	5.0	1.5	0.9	5.0	5.0	5.0	5.0	9.0	9.2	5.0	5.0	10.0	10.0	5.0	5.0	5.0	5.0	5.0
Electricals	13.6	16.7	18.1	21.7	20.5	19 9	20.9	25.0	22.1	22.4	22.9	29.6	21.1	20.1	20.7	25.5	20.4	20.0	20.0	26.9	28.6	27.7	28.0	26.1	25.0	28.7	28.9	25.0
Transport	0	0	0	5.0	0	0	0	0.4	0	0	0	0	0	0	0	0	0	0	0	0	4.9	4.5	4.5	5.0	10.0	10.0	10.0	10.0
Retail	27.0	25 0	25.0	25.0	27.1	23.2	22.5	25.0	25.7	23.8	22.8	21.4	23.5	19.7	18.3	16.3	14.3	10.0	10.0	10.0	12.0	10.0	10.0	15.0	10.0	10.0	10.0	10.0

NOTE.—1 = Single-index model; 2 = multi-index model, covariance form; 3 = multi-index model, diagonal form; 4 = Markowitz E.V. model.

* Oil, rubber, stone, glass, and clay.

TABLE 7

COMPARISON OF EFFICIENT PORTFOLIOS BY INDUSTRY GROUPING, 150-SECURITY UNIVERSE

A. NUMBER OF SECURITIES IN EACH INDUSTRY

INDUSTRY	No.	14.25%				16.0%				18.0%				20.0%			
		1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Mines	10	2	2	2	1	1	2	2	1	2	2	2	1	2	1	1	2
Foods	23	8	12	8	6	9	8	8	6	9	7	7	7	8	7	7	6
Textiles	5	1	2	2	2	1	1	1	1	1	1	1	1	1	0	0	1
Paper	9	2	2	2	3	2	2	2	3	2	2	2	2	1	1	1	1
Chemicals	16	7	5	5	4	9	5	5	6	8	6	6	6	7	6	6	6
ORSGC*	16	3	5	5	4	4	5	5	4	4	4	4	3	4	5	6	5
Metals	14	2	3	3	3	2	3	3	3	2	2	3	4	2	2	2	4
Electrical	26	5	5	6	6	7	7	7	7	7	7	8	8	7	9	9	7
Transport	14	1	2	2	3	2	2	2	2	2	3	3	2	1	3	3	2
Retail	17	15	11	11	11	15	12	11	11	14	11	10	11	11	9	9	9
No. in portfolio	.....	46	49	46	43	52	47	46	44	51	45	46	45	44	43	44	43
No. at upper bound	.....	30	34	34	37	33	35	35	38	33	30	33	38	35	37	37	37
% at upper bound	.....	65.2	69.4	73.9	86.0	63.5	74.5	76.1	86.4	64.7	66.7	71 8	84.4	79.6	86.0	84.1	86.0

INDUSTRY	No.	22 0%				24.0%				26.0%				28 0%			
		1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Mines	10	2	2	2	2	2	2	2	3	2	2	2	4	3	4	4	3
Foods	23	7	7	7	5	4	5	5	4	5	5	5	5	4	4	4	4
Textiles	5	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1
Paper	9	2	2	2	1	2	2	2	2	2	2	2	2	3	3	3	3
Chemicals	16	6	5	5	6	6	5	5	6	5	5	5	6	3	2	2	3
ORSGC*	16	5	7	7	6	7	8	8	7	6	6	6	6	6	6	6	6
Metals	14	2	2	2	4	2	3	3	2	2	3	4	2	3	3	3	3
Electrical	26	8	8	9	7	9	9	9	10	9	11	11	9	10	11	11	10
Transport	14	1	2	2	1	1	1	2	2	3	3	3	3	5	4	4	5
Retail	17	10	10	7	9	8	7	7	7	6	5	5	5	3	4	4	3
No. in portfolio	.....	44	46	43	42	42	42	43	44	41	42	43	43	41	41	41	41
No. at upper bound	.....	37	37	38	37	38	38	38	38	39	36	35	37	39	39	38	39
% at upper bound	.....	84.1	80.4	88.4	88.1	90.5	90.5	88.4	86.4	95.2	85.8	81.4	86.0	95.2	95.2	92.7	95.2

B. PERCENTAGE OF PORTFOLIO INVESTED

INDUSTRY	14.25%				16.0%				18.0%				20.0%			
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Mines	3.6	5.0	5.0	2.5	2.5	5.0	5.0	2.5	2.5	4.4	5.0	2.5	3.1	2 5	2.5	3.4
Foods	18.0	19.6	16.8	13.8	17 8	18.4	15.6	12.9	18.9	16.4	15.6	15.6	17.6	15.7	15.1	14.6
Textiles	2.2	3.5	3.7	3.5	2.5	2.5	2 5	2.5	2.5	1.7	1.6	2.5	2.5	0	0	2.5
Paper	5.0	5.0	5.0	7.5	5.0	5.0	5.0	5.6	2.7	3.6	3.4	3.2	2.5	2.5	2.5	2.5
Chemicals	13.6	10.4	9 2	8.4	12.0	12 0	11 8	11.3	13.9	14 6	14.0	13.6	16.3	15.0	15 0	15.0
ORSGC*	7.3	8.2	9.9	9.9	5.2	8.6	9.9	9.1	6.3	9 6	10 0	7.5	8.6	11 3	11.9	9.9
Metals	5 0	5.6	6.5	7.5	4.5	5.1	5.9	7.5	4.0	5 0	5.2	8.6	3.5	5.0	5.0	10.0
Electrical	9.2	10.4	11.4	11.8	12.4	13 1	14.1	16.2	15.0	14 4	15.5	17.8	16.4	18.2	19.0	16.5
Transport	0.8	5.0	5.0	7.5	2 4	5.0	5.0	5.0	2.7	5.4	6.2	5.0	2.5	7.5	7.5	5.0
Retail	35.2	27.5	27.5	27.5	35.8	25.4	25.2	27.5	31.6	24.9	23.4	23.5	27.0	22.4	21.5	20.6

INDUSTRY	22.0%				24.0%				26.0%				28.0%			
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Mines	5.0	2.9	3.8	5.0	5.0	5.0	5 0	5.9	5.0	5.0	5.0	7.1	7.5	8.6	8.8	7 5
Foods	13.1	15.2	15.5	10.9	10.0	12.5	12.5	10.0	10.5	12.5	12 5	10 4	10 0	10 0	10 0	10.0
Textiles	2.5	0.1	0	2.5	2.5	0	0	2.5	2 5	0	0	2 5	2.5	0	0	1.2
Paper	3.4	5.0	5.0	2.5	5.0	5.0	5.0	5.0	5 0	5.0	5.0	5.0	7 5	7.5	7 5	7.5
Chemicals	15.0	12.5	12.5	15.0	14.6	12.5	12 5	15.0	12.5	11.9	11.7	13.7	6.2	5.0	5.0	7.5
ORSGC*	12.5	17.5	17.5	14.8	17 5	19.7	19.8	15.9	15.0	15 0	15.0	15 0	15 0	15.0	15.0	15.0
Metals	3.3	5.0	5.0	9.0	5.0	7.5	7.5	5.0	5.0	7.1	8 1	5.0	7.5	7 5	7.5	7.5
Electrical	17.9	17.8	18.2	17.5	19.3	18.9	19.0	22 1	22.5	24.0	23 4	22 5	25 0	27 5	27.1	25.0
Transport	2.5	4.8	5.0	2.5	2.5	2.5	2.6	3.0	7.0	7 0	6 8	7.5	11.3	10 0	10.0	11.3
Retail	24.7	19.3	17.5	20.3	18.6	16.4	16.1	15.6	15.0	12.5	12.5	11.2	7.5	8.9	9.2	7.5

NOTE.—1 = single-index model; 2 = multi-index model, covariance form; 3 = multi-index model, diagonal form; 4 = Markowitz E.V. model.

* Oil, rubber, stone, glass, and clay.

parison. The general pattern of relationships shown also holds true for similar comparisons using any of the other three models. The other graphs have not been included, however, for the sake of brevity.

From Figures 4 and 5 it is observed that the ex post results of the efficient portfolios tend to be grouped together. This is due in part to the naïveté of the method by which expectations about security returns were formed. In addition, the ex post results tend to be grouped further from the efficient frontier when the security universe size is increased.

As noted from Figures 4 and 5, the

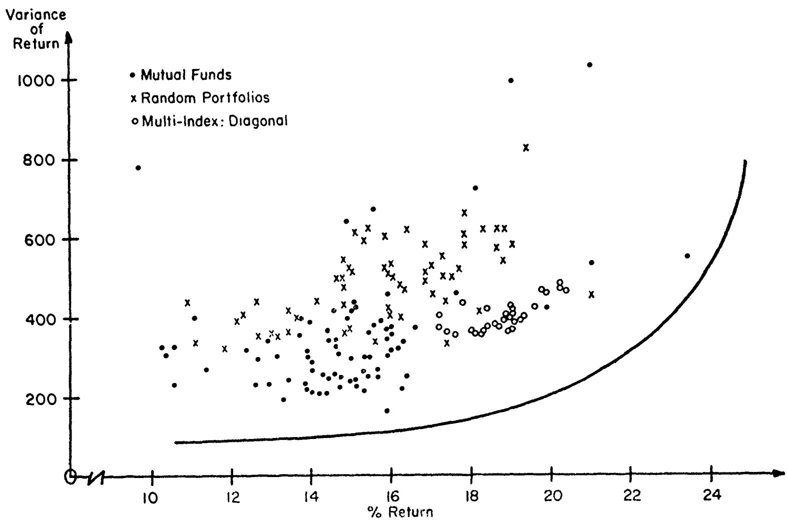

FIG. 4.—Comparison of ex post performance of 78 mutual funds and 60 randomly selected portfolios with the multi-index (diagonal form) model: 75-security universe.

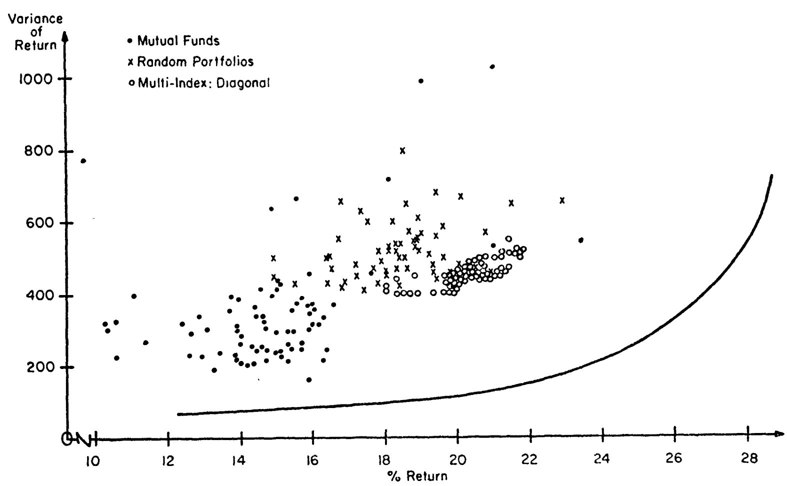

FIG. 5.—Comparison of ex post performance of 78 mutual funds and 60 randomly selected portfolios with the multi-index (diagonal form) model: 150-security universe.

portfolios selected by the diagonal multi-index model tend to dominate the random portfolios. The mutual funds, however, are *not* dominated ex post by either the random- or model-selected portfolios, but tend to be more conservative, accepting less risk and a lower return. It should be pointed out that the mutual fund returns have not been corrected for management fees; hence they appear in a standard deviation of return has increased from 20.0 per cent to 21.4 per cent.

The lines drawn through the two groups of points represent risk-return trade-off relationships. The slopes of these lines define the increase in risk that must be accepted to increase return by 1 per cent.[15]

Figure 7 shows the risk-return trade-

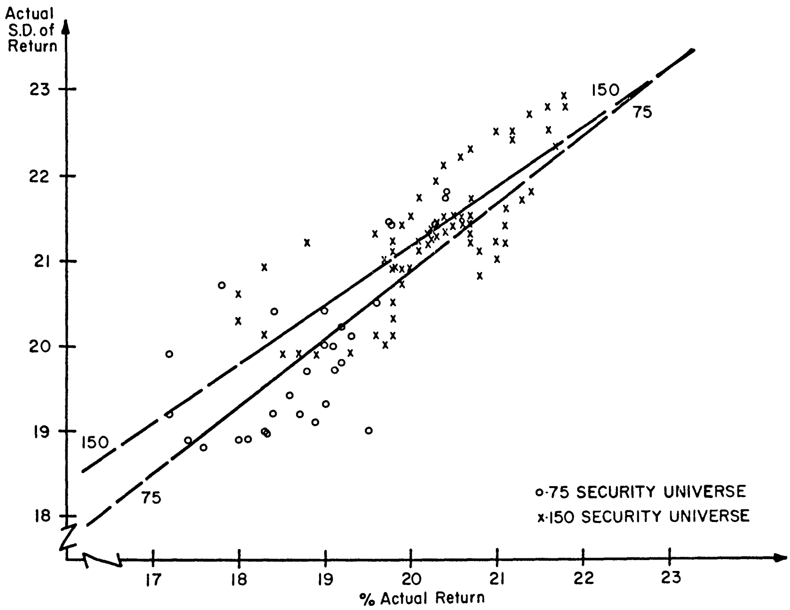

FIG. 6.—Ex post performance, multi-index model (diagonal form)

slightly less favorable light. Correction for the management fee (which is approximately 0.5–1.0 per cent per annum) would tend to shift the distribution of mutual funds in the direction of higher ex post return.

Figure 6 shows the ex post results for the multi-index model (diagonal form) for both security universe sizes. It is seen that the levels of both ex post risk and return have increased for the larger universe size. The average ex post return per portfolio has increased from 18.8 per cent to 20.3 per cent, while the average

[15] A statistical comment should be made with regard to these risk-return relationships. It is possible to obtain a positive relationship between sample portfolio mean returns and variances of returns even though all of the samples were drawn from the same underlying density function $f(R_1, \ldots, R_N)$. This would result if the single underlying distribution was positively skewed, in which case the covariance of sample means and variances would equal the third population moment about the mean divided by the number of observations in each sample (i.e., $COV\ [m_1, m_2] = \mu_3/N$). If this were the case, then the slope coefficient observed would only be a function of the skewness of the underlying distribution.

Evidence from various researchers has indicated that security-yield distributions tend to be positively skewed. In this case, the above statistical problems would arise in attempting to correlate the mean annual returns of different securities with the variances off relationships for each of the models considered for each universe size. In addition, the diagram shows the relationships inherent in the ex post performance of the two groups of random portfolios and the group of mutual funds. The ordering of these lines indicates dominance relationships among the ex post performances of various methods of portfolio selection. However, while its lines appear distinct in the diagram, it is not obvious whether the observed orderings are statistically significant. This is due to the incomplete fulfilment by the ex post data of various statistical assumptions implicit to the linear regression model.

When statistical tests are made to determine the significance of the observed differences among the regression lines, the following conclusions can be drawn:

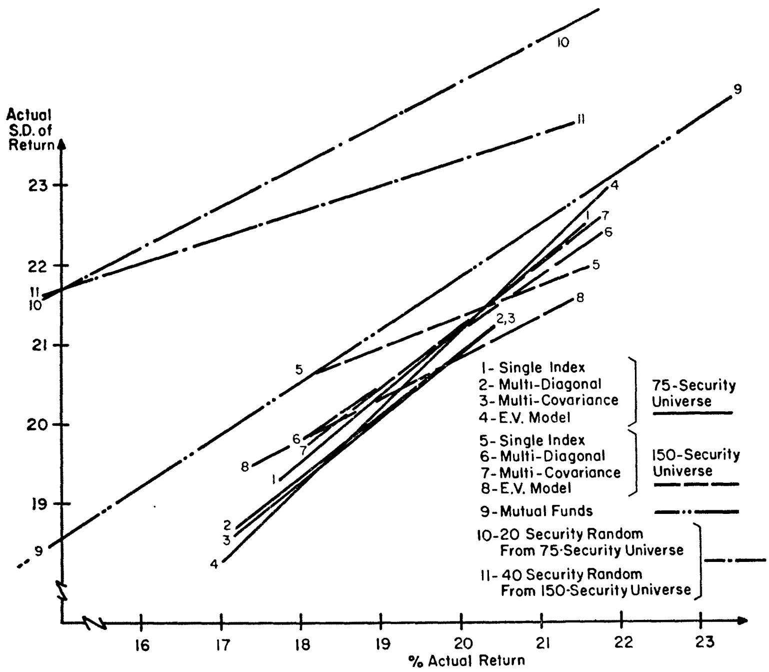

FIG. 7.—Comparison of ex post performance risk-return trade-off relationships

of their annual returns. In our case, however, we are dealing with portfolio-yield distributions rather than security-yield distributions. Since we have an average of 23 securities in each of the 75-security universe portfolios and 43 securities in each of the 150-securities universe portfolios (see Table 5), the Central Limit Theorem should insure that the portfolio yield distributions are reasonably symmetric. Hence we can conclude that the positive sloping relationships obtained from the ex post performances of the various models are for the most part inherently due to differences in the portfolio-yield distributions $f_i(R_1, \ldots, R_N)$ and not statistical artifacts induced by the skewness of a single underlying distribution $f(R_1, \ldots, R_N)$.

1. The ex post performance of the efficient portfolios selected by the models and the mutual funds clearly dominates that of the random portfolios.

2. The ex post performance of the efficient portfolios tends to dominate the performance of the mutual funds for higher levels of actual return (above 15 per cent).

3. The performances of the mutual funds with returns less than 15 per cent

are not dominated by the efficient portfolios.

4. There is no strong evidence (at the 5 per cent level of significance) for the absolute dominance of any of the portfolio selection models over the total range of returns available. There is a tendency for the Markowitz model to perform most effectively over more restricted ranges, followed by the covariance form of the multi-index model.

V. CONCLUDING REMARKS

On the basis of empirical evidence provided in the study, the single-index model is seen to have more desirable ex ante properties than the more elaborate multi-index formulations. In particular, the ex ante efficient portfolios produced by the single-index model have lower expected risks than those of the multi-index formulations for equivalent levels of return, and the former are computationally less costly to obtain than are the latter. The ex post picture is not clear. The lack of clearer ex post differentiation is due in part to the naïveté of our security model and in part to the fact that only common stocks are included in the universes of securities considered. It thus does not appear worthwhile at this time to devote effort to developing an efficient computational algorithm for one of the multi-index models, if our primary interest is in common-stock universes. When broader universes of securities are considered (e.g., when various types of bonds and preferred stocks are included along with common stocks), it is expected that the richer representation of the variance-covariance matrix permitted by the multi-index models in comparison with the single-index model will become more relevant and more necessary. If such is the case, then, as shown in this paper, the diagonal form of the multi-index model would be the most useful. This model's ex ante and ex post performances are almost identical with the covariance form, while it has more desirable computational properties. The computational requirements for this model are only slightly more complex than those of the Sharpe single-index model. (As with the single-index model, the diagonal multi-index model has a "diagonal" form covariance matrix.)

Despite the admittedly naïve security-evaluation model which provided the input data to the four models considered in this paper, it is encouraging that the ex post performance of the efficient portfolios selected by each of these models clearly dominated the ex post performance of randomly selected portfolios. The actual performance of the mutual funds during the ex post period also clearly dominates the performance of both sets of random portfolios.[16]

[16] In a recently published paper, Friend and Vickers stated: "We conclude, therefore, that there is still no evidence—either in our new or old tests, or in the tests so far carried out by others—that mutual fund performance is any better than that realizable by random or mechanical selection of stock issues" (Irwin Friend and Douglas Vickers, "Portfolio Selection and Investment Performance," *Journal of Finance*, XX, No. 3 [September, 1965], 412). The evidence that we have obtained in our study stands in clear disagreement with their conclusion, even though the time periods considered for the ex post evaluations are almost identical. Friend and Vickers did not use the actual investment returns achieved by mutual funds, as we did; furthermore, their random portfolio returns were based not upon random portfolios of actual common stocks, as were ours, but rather, upon random portfolios invested in composite industry indexes. Since mutual funds in fact reconstitute their portfolios when their managers feel that this will improve investment performance, and since neither mutual funds nor individual investors can in fact invest in composite industry indexes rather than individual securities, we feel that the methods of comparison employed by us are more relevant than those utilized by Friend and Vickers.

In the same paper (p. 413), Friend and Vickers also state: "This paper, in addition, points up the dangers of using past measures of return and variance as a basis for portfolio selection, or of assuming

It is also interesting to note that the ex post performance of the efficient portfolios was not dominated by (and if anything was superior to) the ex post performance of the mutual funds. This result is particularly striking, since the mutual funds were fully managed during the 1958–64 evaluation periods, while the efficient portfolios were unchanged after their initial selection. Furthermore, the mutual funds presumably employed a more sophisticated (and certainly a more expensive) method of security evaluation than the naïve procedure employed by us in generating efficient portfolios. Finally, the mutual funds were able to invest a much broader universe of common stocks than we employed in our analysis.

In the light of all these factors, the results we have obtained definitely suggest that formal models for selecting efficient portfolios must be considered as very relevant components in the development of improved normative procedures for investment management.

that the procedures for portfolio selection outlined by Markowitz provide any clues to future investment performance." We wish to stress that the conclusions we have reached in this study strongly contradict this statement by Friend and Vickers.

APPENDIX

MARKOWITZ MODEL

$$R_p = \mathbf{R}'\mathbf{X}\,.$$

$$E = E(R_p) = \mathbf{E}'\mathbf{X} = \sum_{i=1}^{N} X_i E(R_i)\,.$$

$$V = VAR(R_p) = \mathbf{X}'\mathbf{\Sigma}_N\mathbf{X} = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \sigma_{ij}\,.$$

SINGLE-INDEX MODEL

$$R_i = A_i + B_i I + C_i\,, \qquad i = 1, \ldots, N.$$

$$I = A_{N+1} + C_{N+1}\,.$$

$$R_p = \sum_{i=1}^{N} X_i(A_i + B_i I + C_i)$$

$$= \sum_{i=1}^{N} X_i(A_i + C_i) + \left[\sum_{i=1}^{N} X_i B_i\right] I\,.$$

Let $X_{N+1} = \sum_{i=1}^{N} X_i B_i\,,$

$$\therefore R_p = \sum_{i=1}^{N} X_i(A_i + C_i) + X_{N+1}(A_{N+1} + C_{N+1})$$

$$= \sum_{i=1}^{N+1} X_i(A_i + C_i)\,.$$

Let $\boldsymbol{R}'$ = the $N+1$ security return vector

$$= (A_1 + C_1, A_2 + C_2, \ldots, A_{N+1} + C_{N+1}) ,$$

$$\boldsymbol{X}' = (X_1, \ldots, X_{N+1}) ,$$

$$\boldsymbol{E}' = (A_1, \ldots, A_{N+1}) , \qquad E(C_i) = 0 .$$

$$R_p = \boldsymbol{R}'\boldsymbol{X} ,$$

$$E = E(R_p) = E(\boldsymbol{R}'\boldsymbol{X}) = \boldsymbol{E}'\boldsymbol{X} = \sum_{i=1}^{N+1} X_i A_i .$$

$$V = VAR(R_p)$$

$$= E[\boldsymbol{R}'\boldsymbol{X} - E(\boldsymbol{R}'\boldsymbol{X})]^2$$

$$= E\Big[\sum_{i=1}^{N+1} X_i C_i\Big]^2$$

$$= E\Big[\sum_{i=1}^{N+1} X_i^2 C_i^2\Big] , \qquad \text{as } E(C_i \cdot C_{i'}) = 0, \qquad i \neq i', i = 1, \ldots, N+1$$

$$= \sum_{i=1}^{N+1} X_i^2 Q_i , \qquad \text{as } E(C_i^2) = Q_i$$

$$= \boldsymbol{X}'\boldsymbol{\Sigma}_{N+1}\boldsymbol{X} .$$

Implicit covariance between returns R_i and $R_{i'}$

$$COV(R_i R_{i'}) = E\{[R_i - E(R_i)]\,[R_{i'} - E(R_{i'})]\}$$

$$= E[(B_i C_{N+1} + C_i)\,(B_{i'} C_{N+1} + C_{i'})]$$

$$= E[B_i B_{i'} C_{N+1}^2] + E[C_i C_{i'}] , \qquad \text{as } E(C_{N+1} \cdot C_i) = 0 , \qquad i = 1, \ldots, N$$

$$= B_i^2 Q_{N+1} + Q_i , \qquad \text{if } i = i'$$

$$= B_i B_{i'} Q_{N+1} , \qquad \text{if } i \neq i' , \qquad \text{as } E(C_i C_{i'}) = 0 .$$

MULTI-INDEX MODEL—COVARIANCE FORM

$$R_i = A_i + B_i J_j + C_i , \qquad \{i \mid i \epsilon N_j\} ,$$

where N_j = the set of securities in class j,

$$J_j = A_{N+j} + C_{N+j} , \qquad j = 1, \ldots, M .$$

$$R_p = \sum_{i=1}^{N} X_i R_i$$

$$= \sum_{j=1}^{M} \Big[\sum_{\{i \mid i \epsilon N_j\}} X_i (A_i + B_i J_j + C_i) \Big]$$

$$= \sum_{j=1}^{M} \sum_{\{i \mid i \epsilon N_j\}} X_i (A_i + C_i) + \sum_{j=1}^{M} \Big[\sum_{\{i \mid i \epsilon N_j\}} X_i B_i \Big] J_j .$$

Let $X_{N+j} = \sum_{\{i|i \epsilon N_j\}} X_i B_i$,

$$\therefore R_p = \sum_{j=1}^{M} \sum_{\{i|i \epsilon N_j\}} X_i(A_i + C_i) + \sum_{j=1}^{M} X_{N+j} J_j .$$

Note that the first term on the right-hand side implies summation over all N securities.

$$\therefore R_p = \sum_{i=1}^{N} X_i(A_i + C_i) + \sum_{j=1}^{M} X_{N+j}(A_{N+j} + C_{N+j})$$

$$= \sum_{i=1}^{N+M} X_i(A_i + C_i).$$

$$\text{Let } \mathbf{R}' = (A_1 + C_1, \ldots, A_N + C_N, A_{N+1} + C_{N+1}, \ldots, A_{N+M} + C_{N+M}),$$

$$\mathbf{X}' = (X_1, \ldots, X_N, X_{N+1}, \ldots, X_{N+M}),$$

$$\mathbf{E}' = (A_1, \ldots, A_N, A_{N+1}, \ldots, A_{N+M}),$$

$$\therefore E = E(R_p) = \mathbf{E}'\mathbf{X} = \sum_{i=1}^{N+M} X_i A_i, \qquad \text{as } E(C_i) = 0,\ i = 1, \ldots, N+M.$$

$$V = VAR(R_p)$$

$$= E[\mathbf{R}'\mathbf{X} - E(\mathbf{R}'\mathbf{X})]^2$$

$$= E\left[\sum_{i=1}^{N+M} X_i C_i\right]^2$$

$$= E\left[\sum_{i=1}^{N} X_i^2 C_i^2\right] + E\left[\sum_{j=1}^{M} \sum_{j'=1}^{M} X_{N+j} X_{N+j'} C_{N+j} C_{N+j'}\right],$$

$$\text{as } E(C_{N+j} C_i) = 0,\ i = 1, \ldots, N,\ j = 1, \ldots, M$$

$$= \sum_{i=1}^{N} X_i^2 Q_i + \sum_{j=1}^{M} \sum_{j'=1}^{M} X_{N+j} X_{N+j'} \sigma_{jj'},$$

where $\sigma_{jj'}$ is the covariance between the levels of industry indexes J_j and $J_{j'}$. When $j = j'$, $\sigma_{jj'} = Q_{N+j}$.

$$\therefore V = \mathbf{X}_N' \mathbf{\Sigma}_N \mathbf{X}_N + \mathbf{X}_M' \mathbf{X}_M \mathbf{X}_M$$

$$= \mathbf{X}' \mathbf{\Sigma}_{N+M} \mathbf{X}.$$

Implicit covariance between returns R_i and $R_{i'}$

$$COV(R_i R_{i'}) = E\{[R_i - E(R_i)]\,[R_{i'} - E(R_{i'})]\}$$

$$= E[(B_i C_{N+j} + C_i)\,(B_{i'} C_{N+j'} + C_{i'})]$$

$$= E[B_i B_{i'} C_{N+j} C_{N+j'}] + E[C_i C_{i'}], \qquad \text{as} \quad E(C_{N+j} \cdot C_i) = 0,$$

$$i = 1, \ldots, N, \qquad j = 1, \ldots, M.$$

$$\therefore COV(R_i R_{i'}) = B_i^2 Q_{N+j} + Q_i, \qquad \text{if} \quad i = i'$$

$$= B_i B_{i'} \sigma_{jj'}, \qquad \text{if } i \neq i', \qquad \text{as} \quad E(C_i C_{i'}) = 0, \qquad i = 1, \ldots, N.$$

MULTI-INDEX MODEL—DIAGONAL FORM

In this model an additional index is used as a medium for expressing the relationship between the industry indexes.

$$J_j = A_{N+j} + B_{N+j}I + C_{N+j}\,, \qquad j = 1, \ldots, M\,,$$

where $\qquad I = A_{N+M+1} + C_{N+M+1}\,.$

As in the covariance form of the multi-index model,

$$R_p = \sum_{i=1}^{N} X_i(A_i + C_i) + \sum_{j=1}^{M} X_{N+j}J_j\,.$$

Consider the second term on the right-hand side, and let $J_j = A_{N+j} + B_{N+j}I + C_{N+j}$,

$$\therefore \sum_{j=1}^{M} X_{N+j}J_j = \sum_{j=1}^{M} X_{N+j}(A_{N+j} + B_{N+j}I + C_{N+j})$$

$$= \sum_{j=1}^{M} X_{N+j}(A_{N+j} + C_{N+j}) + \left[\sum_{j=1}^{M} X_{N+j}B_{N+j}\right]I\,.$$

Let $X_{N+M+1} = \sum_{j=1}^{M} X_{N+j}B_{N+j}$,

$$\therefore \sum_{j=1}^{M} X_{N+j}J_j = \sum_{j=1}^{M+1} X_{N+j}(A_{N+j} + C_{N+j})\,.$$

$$\therefore R_p = \sum_{i=1}^{N} X_i(A_i + C_i) + \sum_{j=1}^{M+1} X_{N+j}(A_{N+j} + C_{N+j})\,,$$

$$R_p = \sum_{i=1}^{N+M+1} X_i(A_i + C_i)\,.$$

Let
$$R' = (A_1 + C_1, \ldots, A_N + C_N, A_{N+1} + C_{N+1}, \ldots, A_{N+M+1} + C_{N+M+1})\,,$$
$$X' = (X_1, \ldots, X_N, X_{N+1}, \ldots, X_{N+M}, X_{N+M+1})\,,$$
$$E' = (A_1, \ldots, A_N, A_{N+1}, \ldots, A_{N+M}, A_{N+M+1})\,.$$

$$\therefore E = E(R_p) = E'X = \sum_{i=1}^{N+M+1} X_iA_i, \qquad \text{as} \qquad E(C_i) = 0, i = 1, \ldots, N+M+1\,.$$

$$\begin{aligned} V &= VAR(R_p) \\ &= E[R'X - E(R'X)]^2 \\ &= E\left(\sum_{i=1}^{N+M+1} X_iC_i\right)^2 \\ &= \sum_{i=1}^{N+M+1} X_i^2Q_i\,, \quad \text{as} \quad E(C_iC_i) = 0\,, \quad i \neq i' \; i = 1, \ldots, N+M+1\,, \\ &= X'\Sigma_{N+M+1}X\,. \end{aligned}$$

Implicit covariance between returns R_i and $R_{i'}$

$$
\begin{aligned}
COV(R_i R_{i'}) &= E\{[R_i - E(R_i)]\,[R_{i'} - E(R_{i'})]\} \\
&= E[(B_i\{B_{N+j}C_{N+M+1} + C_{N+j}\} + C_i)\,(B_{i'}\{B_{N+j'}C_{N+M+1} + C_{N+j'}\} + C_{i'})] \\
&= B_i B_{i'} B_{N+j} B_{N+j'} Q_{N+M+1}\,, \quad \text{if} \quad i \neq i' \quad \text{and} \quad j \neq j' \\
&= B_i B_{i'} B^2{}_{N+j} Q_{N+M+1} + B_i B_{i'} Q_{N+j}\,, \quad \text{for} \quad i \neq i' \quad \text{and} \quad j = j'\,. \\
&= B_i^2 B^2{}_{N+j} Q_{N+M+1} + B_i^2 Q_{N+j} + Q_i\,, \quad \text{for} \quad i = i' \quad \text{and} \quad j = j'\,.
\end{aligned}
$$