Problem 1  (20 points)

Answer the following questions:

(a) Suppose the single index model holds. Show that the cut-off point $C^*$ can be written as:

$$
C^* = (R_p - R_f) \beta_p \frac{\sigma_m^2}{\sigma_p^2}.
$$

$$
C^* = \sigma_m^2 \sum \hat{x}_i \beta_j = \sigma_m^2 \sum \lambda \chi \beta_j.
$$

$$
= \sigma_m^2 \sum \frac{R_p - R_f}{\sigma_p^2} \chi \beta_j = \sigma_m^2 \frac{\overline{R_p - R_f}}{\sigma_p^2} \beta_p.
$$

$$
\Rightarrow C^* = \left( \frac{\overline{R_p - R_f}}{\sigma_p^2} \beta_p \right) \frac{\sigma_m^2}{\sigma_p^2}.
$$

(b) Assume the constant correlation model holds and that short sales are allowed. Under what condition

the cut-off point $C^*$ is equal to:

$$
C^* = \frac{1}{N} \sum_{i=1}^{N} \frac{\overline{R_i} - R_f}{\sigma_i}.
$$

$$
C^* = \frac{\rho}{1 - \rho + N \rho} \sum \frac{\overline{R_i} - R_f}{\sigma_i} = \frac{\rho}{1 - \rho + N \rho} \sum \frac{\overline{R_i} - R_f}{\sigma_i}.
$$

$$
= \frac{1}{\rho - 1 + N} \left\{ \sum \frac{\overline{R_i} - R_f}{\sigma_i} \right\} \Rightarrow C^* = \frac{1}{N} \sum \frac{\overline{R_i} - R_f}{\sigma_i}.
$$
Problem 2 (25 points)

You are given the following data:

<table>
<thead>
<tr>
<th>Stock i</th>
<th>( R_i )</th>
<th>( \sigma_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.29</td>
<td>0.03</td>
</tr>
<tr>
<td>2</td>
<td>0.19</td>
<td>0.02</td>
</tr>
<tr>
<td>3</td>
<td>0.08</td>
<td>0.15</td>
</tr>
</tbody>
</table>

a. Assume short sales are allowed, \( R_f = 0.05 \), and \( \rho = 0.5 \). Rank the stocks based on the excess return to standard deviation ratio, find the cut-off point \( C^* \), and find the optimum portfolio.

\[
\frac{R_i - R_f}{\sigma_i} = \frac{0.5}{1 - 0.5 + 0.5} = 0.5, \quad \text{Stock 1:} 8
\]
\[
\frac{0.19 - 0.05}{0.02} = 7, \quad \text{Stock 2:} 15
\]
\[
\frac{0.08 - 0.05}{0.15} = 0.2, \quad \text{Stock 3:} 15.2
\]

\[
\frac{3}{(1 - 0.5)^{0.03}} \left[ \frac{8 - 3.8}{3.8} \right] = 28.0, \quad \text{Stock 1:} 28.0
\]
\[
\frac{3}{(1 - 0.5)^{0.02}} \left[ \frac{7 - 3.8}{3.8} \right] = 32.0, \quad \text{Stock 2:} 32.0
\]
\[
\frac{1}{(1 - 0.5)^{0.05}} \left[ \frac{12 - 3.8}{3.8} \right] = -4.8, \quad \text{Stock 3:} -4.8
\]

\[
\chi_1 = \frac{28.0}{3.8} = 7.368, \quad \chi_2 = \frac{32.0}{3.8} = 8.421
\]

\[
\chi_3 = \frac{-4.8}{3.8} = -1.263
\]

b. The above solution could have been found using the techniques that discussed earlier in class through the following:

\[
Z = \Sigma^{-1} R = \begin{pmatrix}
0.00099 & 0.00030 & 0.00225 \\
0.00030 & 0.00040 & 0.00150 \\
0.00225 & 0.00150 & 0.02250
\end{pmatrix}^{-1} \begin{pmatrix}
0.29 - 0.05 \\
0.19 - 0.05 \\
0.08 - 0.05
\end{pmatrix} = \begin{pmatrix}
280.00 \\
320.00 \\
-48.00
\end{pmatrix}
\]

Explain what you see here and verify that the solution is the same as with part (a) (there may be some rounding differences).

\[
\sum \chi_i = \sum \frac{Z_i}{\sigma_i} = 5 \sum \frac{Z_i}{\sigma_i} = 5 \chi
\]

\[
\chi_1 = 280/3.8 = 7.368
\]

\[
\chi_2 = 320/3.8 = 8.421
\]

\[
\chi_3 = -48/3.8 = -1.263
\]
Problem 3 (30 points)
Suppose the single index model holds. Also short sales are allowed and there is a risk free rate \( R_f = 0.002 \). For 3 stocks the following were obtained based on monthly returns for a period of 5 years.

<table>
<thead>
<tr>
<th>Stock i</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \sigma^2_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
<td>1.08</td>
<td>0.003</td>
</tr>
<tr>
<td>2</td>
<td>0.04</td>
<td>0.50</td>
<td>0.006</td>
</tr>
<tr>
<td>3</td>
<td>0.08</td>
<td>1.22</td>
<td>0.001</td>
</tr>
</tbody>
</table>

The expected return and variance of the market are \( \bar{R}_m = 0.10 \) and \( \sigma^2_m = 0.002 \) for the same period.

a. Suppose that the optimum portfolio consists of 30% of stock 1, 50% of stock 2, and 20% of stock 3. What is the \( \beta \) of this portfolio?

\[
\beta_P = \sum \beta_i \alpha_i = 0.30 (1.08) + 0.50 (0.80) + 0.20 (1.12) = 1.70
\]

\[
\bar{R}_P = \beta_P \bar{R}_m = 1.70 (0.10) = 0.17
\]

\[
\sigma_P = \sqrt{\sum \beta_i^2 \sigma^2_i} = \sqrt{(0.30)^2 (0.003) + (0.50)^2 (0.006) + (0.20)^2 (0.001)} = 0.0579
\]

b. Suppose that you are a portfolio manager and you have $300,000 to invest in this optimum portfolio on behalf of a client. In addition this client wants to invest another $300,000 by borrowing this amount at the risk free rate \( R_f = 0.002 \). What is the expected return and standard deviation of this portfolio. Show it on the expected return standard deviation space.

\[
\bar{R}_c = (1 - \lambda) \bar{R}_P + \lambda \bar{R}_f = 0.60 (0.002) + 1.6 \left( 0.30 (0.10) + 0.50 (0.04) + 0.20 (0.02) \right)
\]

\[
\sigma_c^2 = \beta_P^2 \sigma_m^2 = 1.70^2 (0.002) = 0.00568
\]

\[
\sigma_c = \sqrt{\sigma_c^2} = 0.0309
\]

c. What is the covariance between the portfolio of part (a) and the market?

\[
\text{Cov}(0.30 R_1 + 0.50 R_2 + 0.20 R_3, R_m) = 0.30 \text{Cov}(R_1, R_m) + 0.50 \text{Cov}(R_2, R_m) + 0.20 \text{Cov}(R_3, R_m)
\]

\[
= 0.30 \left[ 1.08 \times 0.002 + 0.50 \times 0.006 \right] + 0.20 \left[ 1.12 \times 0.002 \right] = 0.001936
\]

d. If the client wants to allocate 60% of his initial funds in the optimum portfolio and the remaining 40% in the risk free asset, what would be the expected return and standard deviation of this position?

\[
\bar{R}_c = 0.6 \bar{R}_P + 0.4 \bar{R}_f = 0.6 (0.17) + 0.4 (0.002) = 0.108228
\]

\[
\sigma_c = 0.6 \sigma_P = 0.6 (0.0309) = 0.018542
\]

e. What is the covariance between stock 1 and the market?

\[
\text{Cov}(R_1, R_m) = \text{Cov}(\alpha + \beta R_m + \epsilon, R_m)
\]

\[
= \beta \sigma_m^2 = 1.08 \times (0.002) = 0.00216
\]
Problem 4 (25 points)

Assume that $\sigma_m^2 = 10, R_f = 0.05$. You are also given $\beta_1 = 1, \beta_2 = 1.5, \beta_3 = 1, \beta_4 = 2, \beta_5 = 1, \beta_6 = 1.5, \beta_7 = 2, \beta_8 = 0.8, \beta_9 = 1, \beta_{10} = 0.6$. The table below shows the procedure for finding the cut-off point $C^*$.

<table>
<thead>
<tr>
<th>Stock $i$</th>
<th>$\frac{R_i - R_f}{\beta_i}$</th>
<th>$\frac{(R_i - R_f)\beta_i}{\sigma_i^2}$</th>
<th>$\sum_{j=1}^{i} \frac{(R_i - R_f)\beta_j}{\sigma_j^2}$</th>
<th>$\frac{\beta_i^2}{\sigma_i^2}$</th>
<th>$\sum_{j=1}^{i-1} \frac{\beta_j^2}{\sigma_j^2}$</th>
<th>$C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.0</td>
<td>0.20</td>
<td>$\frac{1}{0.02}$</td>
<td>0.02000</td>
<td>0.02000</td>
<td>1.67</td>
</tr>
<tr>
<td>2</td>
<td>8.0</td>
<td>0.45</td>
<td>0.35</td>
<td>0.05625</td>
<td>0.07625</td>
<td>3.69</td>
</tr>
<tr>
<td>3</td>
<td>7.0</td>
<td>0.35</td>
<td>1.00</td>
<td>0.05000</td>
<td>0.12500</td>
<td>4.42</td>
</tr>
<tr>
<td>4</td>
<td>6.0</td>
<td>2.40</td>
<td>3.65</td>
<td>0.40000</td>
<td>0.52625</td>
<td>5.43</td>
</tr>
<tr>
<td>5</td>
<td>6.0</td>
<td>0.15</td>
<td>3.85</td>
<td>0.02500</td>
<td>0.55125</td>
<td>5.36</td>
</tr>
<tr>
<td>6</td>
<td>4.0</td>
<td>0.30</td>
<td>4.15</td>
<td>0.07500</td>
<td>0.62625</td>
<td>4.91</td>
</tr>
<tr>
<td>7</td>
<td>3.0</td>
<td>0.30</td>
<td></td>
<td>0.10000</td>
<td>0.10000</td>
<td>5.02</td>
</tr>
<tr>
<td>8</td>
<td>2.5</td>
<td>0.10</td>
<td>4.25</td>
<td>0.04000</td>
<td>0.76625</td>
<td>4.91</td>
</tr>
<tr>
<td>9</td>
<td>2.0</td>
<td>0.10</td>
<td>4.35</td>
<td>0.05000</td>
<td>0.81625</td>
<td>4.75</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>0.06</td>
<td>4.41</td>
<td>0.06000</td>
<td>0.87625</td>
<td>4.52</td>
</tr>
</tbody>
</table>

a. Find the three missing values.

A: $0.20$

B: $0.72625$

$$C_i = \frac{\sigma_m^2}{1 + \sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_i^2}} = \frac{0.10 \times 3.55}{1 + 0.10 \times (0.05125)} \Rightarrow C_i = 5.451$$

b. If short sales are not allowed find the cut-off point $C^*$ and the composition of the optimum portfolio.

$$C^* = \bar{\beta} = \frac{\beta_i}{\sigma_i^2} (10 - 5.451) = \frac{1}{50} (10 - 5.451)$$

$$\frac{\beta_i^2}{\sigma_i^2} = 0.02 \Rightarrow \frac{1}{\sigma_i^2} = 0.02 \Rightarrow \sigma_i^2 = 50$$

$$\Rightarrow 2_i = 0.091$$

c. If short sales are allowed find the cut-off point $C^*$ and the composition of the optimum portfolio.

$$C^* = 4.52$$

$$2_i = \frac{1}{50} \left[ 10 - 4.52 \right] \Rightarrow 2_i = 0.1096$$

d. Find the correlation coefficient between stock 1 and the market.

$$\rho = \frac{Cov(R_1, R_m)}{\sigma_1 \sigma_m} = \frac{1 \times 0.1}{\sqrt{1 \times 0.1 + 50} \sqrt{10} \Rightarrow \rho = 0.908}$$