Exercise


The Black-Scholes-Merton formula for the value $C$ of a European call option at time $t$ and expiration time at time $T$ is given by

$$C = S_0 \Phi(d_1) - \frac{E}{e^{r(T-t)}} \Phi(d_2)$$

$$d_1 = \frac{ln(S_0/E) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = \frac{ln(S_0/E) + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}$$

Answer the following questions:

1. Find $\Phi'(d_1)$.
2. Show that $S_0 \Phi'(d_1) = \frac{E}{e^{r(T-t)}} \Phi'(d_2)$.
3. Find $\frac{\partial d_1}{\partial S}$ and $\frac{\partial d_2}{\partial S}$.
4. Show that

   $$\frac{\partial C}{\partial t} = -rEe^{-r(T-t)} \Phi(d_2) - S_0 \Phi'(d_1) \frac{\sigma}{2\sqrt{T-t}}.$$

5. Show that $\frac{\partial C}{\partial S} = \Phi(d_1)$.
6. Show that $C$ satisfies the Black-Scholes-Merton differential equation.
7. Show that $C$ satisfies the boundary conditions for a European call option, $C = max(S-E, 0)$ as $t \to T$.

Exercise 2

Assume that a non-dividend-paying stock has an expected return of $\mu$ and volatility of $\sigma$. A financial institution has just announced that it will trade a security that pays off a dollar amount equal to $ln(S_T)$ at time $T$, where $S_T$ denotes the value of the stock price at time $T$. Answer the following questions:

a. Use risk-neutral valuation to calculate the price of the security at time $t$ in terms of the stock price at time $T$.

b. Confirm that your price satisfies the Black-Scholes-Merton differential equation.
1. Since $\Phi(d_1)$ is the cumulative probability that a standard normal random variable is less than $d_1$, i.e.,

$$P(Z \leq d_1) \text{ it follows that } \Phi'(d_1) = \frac{\partial \Phi(d_1)}{\partial d_1} = f(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\ln \frac{S_0}{E} + (r + \frac{1}{2} \sigma^2)(T-t)\right)}{\sigma \sqrt{T-t}}}.$$

2. Because

$$d_2 = \frac{\ln \left(\frac{S_0}{E}\right) + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t},$$

it follows that $d_1 = d_2 + \sigma \sqrt{T-t}$. Therefore,

$$\Phi'(d_1) = \frac{\partial \Phi(d_1)}{\partial d_1} = f(d_1) = f(d_2 + \sigma \sqrt{T-t}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(d_2 + \sigma \sqrt{T-t}\right)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left(d_2^2 + 2d_2 \sigma \sqrt{T-t} + \sigma^2 (T-t)\right)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} d_2^2} \times e^{-d_2 \sigma \sqrt{T-t} - \frac{1}{2} \sigma^2 (T-t)} = \Phi'(d_2) \times e^{-\frac{1}{2} d_2^2} \times e^{-d_2 \sigma \sqrt{T-t} - \frac{1}{2} \sigma^2 (T-t)}$$

but

$$d_2 = \frac{\ln \left(\frac{S_0}{E}\right) + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}},$$

then

$$\Phi'(d_1) = \Phi'(d_2) \times e^{\frac{-\left(\ln \frac{S_0}{E} + (r - \frac{1}{2} \sigma^2)(T-t)\right)}{\sqrt{T-t} - \frac{1}{2} \sigma^2 (T-t)}} = \Phi'(d_2) \times e^{\ln \frac{S_0}{E} - r(T-t)} = \Phi'(d_2) \times e^{\frac{\ln \frac{S_0}{E} - r(T-t)}{S_0}} e^{-r(T-t)} \Phi'(d_2).$$

So, $\Phi'(d_1) = Ge^{-r(T-t)} \Phi'(d_2)$.

3. Use $d_1 = \frac{\ln \left(\frac{S_0}{E}\right) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$ and write it as $d_1 = \frac{\ln S_0 + \ln E + (r + \frac{1}{2} \sigma^2)(T-t)}{S \sigma \sqrt{T-t}}$. Therefore, $\frac{\partial d_1}{\partial S} = \frac{1}{S \sigma \sqrt{T-t}}$. Similarly, because $d_2 = d_1 - \sigma \sqrt{T-t}$, it follows that $\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{S \sigma \sqrt{T-t}}$.

4. Use the B-S-M formula:

$$C = S_0 \Phi(d_1) - \frac{E}{e^{r(T-t)}} \Phi(d_2)$$

$$\frac{\partial C}{\partial t} = S \Phi'(d_1) \frac{\partial d_1}{\partial t} - \frac{r E \Phi(d_2)}{e^{r(T-t)}} - \frac{E}{e^{r(T-t)}} \Phi'(d_2) \frac{\partial d_2}{\partial t}$$

From (2) $S \Phi'(d_1) = e^{-r(T-t)} \Phi'(d_2)$, therefore

$$\frac{\partial C}{\partial t} = S \Phi'(d_1) \frac{\partial d_1}{\partial t} - \frac{r E \Phi'(d_2)}{e^{r(T-t)}} - S \Phi'(d_1) \frac{\partial d_2}{\partial t}$$

$$= S \Phi'(d_1) \left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right) - \frac{r E \Phi'(d_2)}{e^{r(T-t)}}.$$

But $d_1 - d_2 = \sigma \sqrt{T-t}$, which means $\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = -\frac{\sigma}{\sqrt{T-t}}$.

Finally, $\frac{\partial C}{\partial t} = -r E e^{-r(T-t)} \Phi(d_2) - S \Phi'(d_1) \frac{\sigma}{2 \sqrt{T-t}}$, which is a decreasing function of $t$. 

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5. This is the hedge ratio $\frac{\partial C}{\partial S} = \Phi(d_1)$. Again, begin with the formula for $C$.

$$C = S_0 \Phi(d_1) - \frac{E}{e^{r(T-t)}} \Phi(d_2)$$

$$\frac{\partial C}{\partial S} = \Phi(d_1) + S \Phi'(d_1) \frac{\partial d_1}{\partial S} - \frac{E}{e^{r(T-t)}} \Phi'(d_2) \frac{\partial d_2}{\partial S}$$

From (2) $S_0 \Phi'(d_1) = \frac{E}{e^{r(T-t)}} \Phi'(d_2)$

From (3) $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S \sigma \sqrt{T-t}}$. Therefore

$$\frac{\partial C}{\partial S} = \Phi(d_1) + \frac{E}{e^{r(T-t)}} \Phi'(d_2) \frac{1}{S \sigma \sqrt{T-t}} - \frac{E}{e^{r(T-t)}} \Phi'(d_2) \frac{1}{S \sigma \sqrt{T-t}} = \Phi(d_1).$$

6. $C$ satisfies the B-S-M formula.
   From (5) and (3) it follows that $\frac{\partial^2 C}{\partial S^2} = \Phi'(d_1) \frac{\partial d_1}{\partial S} = \Phi'(d_1) \frac{1}{S \sigma \sqrt{T-t}}$. Therefore,

$$\frac{\partial C}{\partial t} \text{ (from (3))} + rS \frac{\partial C}{\partial S} \text{ (from (5))} + \frac{1}{2} S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

7. Examine what happens as $t \to T$.
   If $S > E$ then $d_1 \to \infty$ and $d_2 \to \infty$ and therefore $\Phi(d_1) \to 1$ and $\Phi(d_2) \to 1$.
   In this case $C \to S - E$.
   If $S < E$ then $d_1 \to -\infty$ and $d_2 \to -\infty$ and therefore $\Phi(d_1) \to 0$ and $\Phi(d_2) \to 0$.
   Now, $C \to 0$.

We see that as $t \to T$, $C \to \max(S - E, 0)$. 