Answer the following questions:

a. The betas of 30 stocks were obtained using simple regression in two successive periods: 2007-12-31 to 2011-12-31 (period 1) and 2011-12-31 to 2016-01-31 (period 2). There are 48 months in each period. Suppose we use the unadjusted betas in the first period as predictions of the betas in the second period. We can then compute the PRESS to evaluate the performance of these unadjusted betas. The following information is obtained from these data:

\[ \sum_{i=1}^{30} \beta_{1i} = 31.31761 \quad \text{Sum of the betas in period 1.} \quad \bar{\beta}_1 = 1.04394 \]
\[ \sum_{i=1}^{30} \beta_{2i} = 29.69676 \quad \text{Sum of the betas in period 2.} \quad \bar{\beta}_2 = 0.98789 \]
\[ \text{Var}(\beta_1) = 0.4558062 \quad \text{Variance of the betas in period 1.} \quad \bar{\sigma}_{\beta_1}^2 = 0.4558062 \]
\[ \text{Var}(\beta_2) = 0.3235921 \quad \text{Variance of the betas in period 2.} \quad \bar{\sigma}_{\beta_2}^2 = 0.3235921 \]
\[ \text{Cov}(\beta_1, \beta_2) = 0.170069 \quad \text{Covariance between the betas in the two periods.} \]


\[ \text{PRESS} = (\bar{A} - \bar{\beta})^T \Sigma \bar{\beta} + (I - \bar{Y} \bar{Y}^T) \Sigma \bar{\beta} \]
\[ r = \frac{\text{Cov}(\epsilon_1, \epsilon_2)}{\Sigma \bar{\beta}^T \Sigma \bar{\beta}} = \frac{0.170069}{0.4558062 \times 0.4558062} = 0.442667 \]
\[ \theta = \frac{\text{Cov}(\epsilon_1, \epsilon_2)}{\text{Var}(\epsilon_1)} = \frac{0.170069}{0.4558062} = 0.3729867 \]
\[ \text{PRESS} = 0.4423036 \]

b. Refer to question (a). Suppose now we use the Vasicek adjustment procedure to adjust the betas in period 1 in order to be better predictions for betas in period 2. Which one of the three components of PRESS do you expect to decrease using the adjustment betas? Explain.
c. Consider the single index model. Show that the covariance between the returns of two stocks as computed by the classical Markowitz method (i.e., \( \text{cov}(R_1, R_2) = \frac{1}{n-1} \sum_{t=1}^{n} (R_{1t} - \overline{R}_1)(R_{2t} - \overline{R}_2) \)) is equal to 
\[ \hat{\beta}_1 \hat{\beta}_2 \text{var}(R_m) + \text{cov}(\epsilon_1, \epsilon_2), \]
where \( n \) is the number of months, \( \hat{\beta}_1 \) is the beta coefficient of the regression of \( R_1 \) on \( R_m \), \( \hat{\beta}_2 \) is the beta coefficient of the regression of \( R_2 \) on \( R_m \), \( \text{var}(R_m) = \frac{1}{n-1} \sum_{t=1}^{n} (R_{mt} - \overline{R}_m)^2 \), and \( \epsilon_1 \) and \( \epsilon_2 \) are the residuals of the two regressions. Please comment on this result from the portfolio point of view.

\[
\text{cov} \left( \overline{R}_1, \overline{R}_2 \right) = \frac{1}{n-1} \sum \left( \overline{R}_{1t} - \overline{R}_1 \right) \left( \overline{R}_{2t} - \overline{R}_2 \right) = \frac{1}{n-1} \sum \left[ \hat{R}_{1t} - \hat{R}_1 + \hat{R}_{1t} - \overline{R}_1 \right] \left[ \hat{R}_{2t} - \hat{R}_2 + \hat{R}_{2t} - \overline{R}_2 \right]
\]

\[= \frac{1}{n-1} \sum \left( \hat{R}_{1t} - \hat{R}_1 \right) \left( \hat{R}_{2t} - \hat{R}_2 \right) + \frac{1}{n-1} \sum \left( \hat{R}_{1t} - \overline{R}_1 \right) \left( \hat{R}_{2t} - \overline{R}_2 \right) + \frac{1}{n-1} \sum \left( \hat{R}_{1t} - \overline{R}_1 \right) \left( \hat{R}_{2t} - \overline{R}_2 \right) \]

\[= \text{cov} \left( \epsilon_1, \epsilon_2 \right) + \hat{\epsilon}_1 \hat{\epsilon}_2 \text{var}(R_m)
\]

\[\text{As I def: } \overline{\epsilon}_1 = 0 \]
\[\overline{\epsilon}_2 = 0
\]

d. In the paper “An Analytic Derivation of the Efficient Frontier,” The Journal of Financial and Quantitative Analysis, Vol. 7, No. 4, Robert Merton gives on page 1854 the proportion of the \( k_{th} \) risky asset held in the frontier portfolio with expected return \( \overline{E} \) by

\[x_k = E \frac{\sum_{j=1}^{m} v_{kj}(CE_i - A) + \sum_{j=1}^{m} v_{kj}(B - AE_j)}{D}, \quad k = 1, \ldots, m.
\]

On the same page it is shown that the expected return of the minimum risk portfolio is \( \overline{E} = \overline{A} \). Using equation (1) above show that the proportion of the \( k_{th} \) risky asset of the minimum risk portfolio is

\[x_k = \frac{\sum_{j=1}^{m} v_{kj}}{\overline{E}}, \quad k = 1, \ldots, m.
\]

\[X_k = \frac{A}{C} \sum V_{Kj} \overline{E}_j - A \frac{A}{C} \sum V_{Kj} + B \sum V_{Kj} - A \sum V_{Kj} \overline{E}_j
\]

\[= \frac{B - \frac{A^2}{C}}{D} \sum V_{Kj} = \frac{B \frac{A^2}{C}}{-BC - A^2} \sum V_{Kj}
\]

\[D = BC - A^2 \]

\[\rightarrow \quad \boxed{X_k = \frac{\sum V_{Kj}}{C}}
\]
e. Consider a portfolio consisting of \( n \) risky assets. Describe three methods to trace out the efficient frontier when short sales allowed.

- **Method 1:**
  - Find Two Portfolios
  - Use Many Risk Values

f. Suppose the single index model holds and short sales are allowed. Two portfolios consisting of the same \( n \) stocks are located on the efficient frontier. Let \( x_1, \ldots, x_n \) be the weights of portfolio \( A \) and \( y_1, \ldots, y_n \) be the weights of portfolio \( B \). Find the covariance between these two portfolios in terms of the betas and variances of the error terms.

\[
\text{Cov}(R_A, R_B) = \text{Cov}\left( \sum \chi_i R_i, \sum \psi_i R_i \right) = \sum \chi_i (\alpha_i + \beta_i \mu + \epsilon_i) \sum \psi_i (\alpha_i + \beta_i \mu + \epsilon_i) = \text{Cov}\left( \sum \chi_i \alpha_i + \beta_i \sum \chi_i \mu + \sum \chi_i \epsilon_i, \sum \psi_i \alpha_i + \beta_i \sum \psi_i \mu + \sum \psi_i \epsilon_i \right) = \text{Cov}(\alpha_{PA} + \beta_{PA} \mu + \sum \chi_i \epsilon_i, \alpha_{PB} + \beta_{PB} \mu + \sum \psi_i \epsilon_i) = \beta_{PA} \beta_{PB} \sigma^2 + \sum \chi_i \psi_i \sigma^2
\]

\( g. \) Refer to question (f). Suppose the single index model holds, short sales are allowed, and risk-free asset exists. Two portfolios \( A \) and \( B \) are located on the capital asset allocation line (CAL). Find the correlation between portfolio \( A \) and \( B \).

\[
\text{Cov}(R_A, R_B) = \text{Cov}\left( x_1 R_1 + (1-x_1) R_f, x_2 R_2 + (1-x_2) R_f \right) \quad \Rightarrow \quad x_1 x_2 \sigma^2
\]

\[
\text{Var}(R_A) = \text{Var}(x_1 R_1 + (1-x_1) R_f) = x_1^2 \sigma^2
\]

\[
\text{Var}(R_B) = \text{Var}(x_2 R_2 + (1-x_2) R_f) = x_2^2 \sigma^2
\]

\[
\text{Cor}(R_A, R_B) = \frac{\text{Cov}(R_A, R_B)}{\sigma_A \sigma_B} = \frac{x_1 x_2 \sigma^2}{\sqrt{x_1^2 \sigma^2} \sqrt{x_2^2 \sigma^2}} = 1.
\]
\[ z = \sum' \bar{R} = \sum' \bar{R} - \bar{R}_f \sum' \bar{1} \]

h. Consider a portfolio of \( n \) risky assets, short sales are allowed, and risk free asset exists. For what value of \( R_f \) the \( k^{th} \) risky asset (a) will not be held, (b) will be held long, and (c) will be held short?

(a) \( z_k = 0 \implies R_f = \frac{\sum V_{kji} \bar{R}_i}{\sum V_{kji}} \)

(b) \( z_k > 0 \implies R_f < \frac{\sum V_{kji} \bar{R}_i}{\sum V_{kji}} \)

(c) \( z_k < 0 \implies R_f > \frac{\sum V_{kji} \bar{R}_i}{\sum V_{kji}} \)

i. Consider a portfolio of \( n = 3 \) risky assets, short sales are allowed, and risk free asset exists, with \( R_f = 0.001 \).

The inverse of the variance covariance matrix and the mean returns of the three stocks are given below:

**Inverse of the variance covariance matrix:**

\[
\begin{bmatrix}
A & B & C \\
A & 325.66 & -96.31 & -348.91 \\
B & -96.31 & 889.84 & -882.07 \\
C & -348.91 & -882.07 & 2213.58
\end{bmatrix}
\]

**Mean returns:**

\[
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix} = \begin{bmatrix}
0.010 \\
0.011 \\
0.014
\end{bmatrix}
\]

Find the composition of the optimal portfolio.

\[
x = \frac{z}{1^t} = \begin{bmatrix}
-0.231^t \\
-0.332 \\
1.555^t
\end{bmatrix}
\]

j. Refer to question (i). Find the mean return of a portfolio that consists of 50% in the risk free asset and 50% in the optimal portfolio found in question (i) and show it on the capital allocation line. Express the standard deviation of this portfolio in terms of \( \sigma_G \) (the standard deviation of the optimal portfolio in question (i)).

\[
\bar{R}_A = \frac{1}{2} \bar{R}_f + \frac{1}{2} \bar{R}_G \\
\bar{R}_G = x'R = \bar{x}' \bar{R} = 0.0159
\]

\[
\bar{R}_A = \frac{1}{2} (0.001) + \frac{1}{2} (0.0159) \approx 0.00845^r
\]

\[
\sigma_A = \frac{1}{2} \sigma_G - \bar{R}_f \sigma = 0.00845^r
\]