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Statistics C183/C283 **Instructor: Nicolas Christou**

Homework 3 - Solutions

Exercise 1

The solution is based on $\mathbf{Z} = \mathbf{\Sigma^{-1} R}$, where, $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$, and $\mathbf{R} = \begin{pmatrix} \bar{R}_A = \bar{R}_1 - R_f \\ \bar{R}_B = \bar{R}_2 - R_f \end{pmatrix}$.

The inverse of $\mathbf{\Sigma}$ is $\mathbf{\Sigma}^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$.

Therefore, $\mathbf{Z} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix} \begin{pmatrix} \bar{R}_A \\ \bar{R}_B \end{pmatrix}$.

It follows that $Z_1 = \frac{\bar{R}_A\sigma_2^2 - \bar{R}_B\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$ and $Z_2 = \frac{-\bar{R}_A\rho\sigma_1\sigma_2 + \bar{R}_B\sigma_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$.

Finally, $X_1 = \frac{Z_1}{Z_1+Z_2} = \frac{\bar{R}_A\sigma_2^2 - \bar{R}_B\rho\sigma_1\sigma_2}{\bar{R}_A\sigma_1^2 + \bar{R}_B\sigma_2^2 - (\bar{R}_A + \bar{R}_B)\rho\sigma_1\sigma_2}$, or $\mathrm{X}_1 = \frac{\bar{R}_A\sigma_2^2 - \bar{R}_B\sigma_{12}}{\bar{R}_A\sigma_1^2 + \bar{R}_B\sigma_2^2 - (\bar{R}_A + \bar{R}_B)\sigma_{12}}$, and $X_2 = 1 - X_1$.

Exercise 2

The solution is based on finding the Z values. Note: You can also solve this problem using the results of exercise 2.

a. Since $X_A = X_B = \frac{1}{2}$ it follows that $Z_A = Z_B$. We get the following system:

$$\begin{aligned} 0.12 - 0.04 &= 0.04Z_A + 0.0016Z_B \\ \bar{R}_B - 0.04 &= 0.0016Z_A + 0.0064Z_B \end{aligned}$$

Solve for Z_A to get $Z_A = 1.9231$. Therefore, $\bar{R}_B = 0.04 + (0.0016 + 0.0064)1.9231 \Rightarrow \bar{R}_B = 0.055385$.

b. Stock B will not be held implies that $X_B = 0$ therefore, $Z_B = 0$.

$$\begin{aligned} 0.12 - 0.04 &= 0.04Z_A + 0.0016Z_B \\ \bar{R}_B - 0.04 &= 0.0016Z_A + 0.0064Z_B \end{aligned}$$

But, $Z_B = 0$, therefore, $Z_A = 2$. It follows that $\bar{R}_B = 0.0432$.

Exercise 4

(a). 1. $0.90 \times 0.0067 + 0.1 \times 0.0125 = 0.00728$

2. $Cov = 0.4(0.0237)(0.0295) = 0.0002797$

3. $SD = \sqrt{0.9^2(0.0237)^2 + 0.1^2(0.0295)^2 + 2(0.9)(0.1)(0.0002797)} = 0.02267$

(b). 1. $0.90(0.0067) + 0.1(0.0042) = 0.00645$

2. $SD = \sqrt{0.9^2(0.0237)} = 0.0213$

Problem 5

(a). $\underset{\sim}{X} = \lambda_1 \underset{\sim}{\Sigma}^{-1} \bar{R} + \lambda_2 \underset{\sim}{\Sigma}^{-1} \underset{\sim}{1}$

WHERE $\lambda_1 = \dfrac{CE-A}{D}$, $\lambda_2 = \dfrac{B-AE}{D}$

THEREFORE, $X = \dfrac{CE-A}{D} \underset{\sim}{\Sigma}^{-1} \underset{\sim}{\bar{R}} + \dfrac{B-AE}{D} \underset{\sim}{\Sigma}^{-1} \underset{\sim}{1}$

$= \dfrac{1}{D}\left(B \underset{\sim}{\Sigma}^{-1} \underset{\sim}{1} - A \underset{\sim}{\Sigma}^{-1} \underset{\sim}{\bar{R}}\right) + \dfrac{1}{D}\left(C \underset{\sim}{\Sigma}^{-1} \underset{\sim}{\bar{R}} - A \underset{\sim}{\Sigma}^{-1} \underset{\sim}{1}\right) E$

$= \underset{\sim}{g} + \underset{\sim}{h} E.$

(b). $R_a = \underset{\sim}{X}_a' \underset{\sim}{R} = (\underset{\sim}{g} + \underset{\sim}{h} E_a)' \underset{\sim}{R}$

$R_b = \underset{\sim}{X}_b' \underset{\sim}{R} = (\underset{\sim}{g} + \underset{\sim}{h} E_b)' R$

$Cov(R_a, R_b) = \underset{\sim}{X}_a' \underset{\sim}{\Sigma} \underset{\sim}{X}_b$

$= (\underset{\sim}{g}' + \underset{\sim}{h}' E_a) \underset{\sim}{\Sigma} (\underset{\sim}{g} + \underset{\sim}{h} E_b)$

$= \underset{\sim}{g}' \underset{\sim}{\Sigma} \underset{\sim}{g} + \underset{\sim}{g}' \underset{\sim}{\Sigma} \underset{\sim}{h} E_b + \underset{\sim}{h}' \underset{\sim}{\Sigma} \underset{\sim}{g} E_a + \underset{\sim}{h}' \underset{\sim}{h} E_a E_b$

$$\mathbf{g}'\Sigma\mathbf{g} = \frac{1}{D^2}\left[B\mathbf{1}'\Sigma^{-1} - A\mathbf{R}'\Sigma^{-1}\right]\Sigma\left[B\Sigma^{-1}\mathbf{1} - A\Sigma^{-1}\mathbf{R}\right]$$

$$= \frac{1}{D^2}\left(B^2\mathbf{1}'\Sigma^{-1}\mathbf{1} - A^2B - A^2B + A^2B\right) = \frac{1}{D^2}\left[B^2C - A^2B\right]$$

$$\mathbf{g}'\Sigma\mathbf{h} = \frac{1}{D^2}\left(B\mathbf{1}'\Sigma^{-1} - A\mathbf{R}'\Sigma^{-1}\right)\Sigma\left(C\Sigma^{-1}\mathbf{R} - A\Sigma^{-1}\mathbf{1}\right)$$

$$= \frac{1}{D^2}\left(BCA - ABC - ABC + A^3\right) = \frac{1}{D^2}\left(A^3 - ABC\right)$$

$$\mathbf{h}'\Sigma\mathbf{g} = \frac{1}{D^2}\left(C\mathbf{R}'\Sigma^{-1} - A\mathbf{1}'\Sigma^{-1}\right)\Sigma\left(B\mathbf{1}'\Sigma^{-1} - A\mathbf{R}'\Sigma^{-1}\right)$$

$$= \frac{1}{D^2}\left(BCA - CAB - ABC + A^3\right) = \frac{1}{D^2}\left(A^3 - ABC\right)$$

$$\mathbf{h}'\mathbf{h} = \frac{1}{D^2}\left(C\mathbf{R}'\Sigma^{-1} - A\mathbf{1}'\Sigma^{-1}\right)\Sigma\left[C\Sigma^{-1}\mathbf{R} - A\Sigma^{-1}\mathbf{1}\right]$$

$$= \frac{1}{D^2}\left(C^2B - A^2C - A^2C + A^2C\right) = \frac{1}{D^2}\left[C^2B - A^2C\right]$$

THEREFORE:

$$\mathrm{Cov}(R_a, R_b) = \frac{1}{D^2}\left[B^2C - A^2B + (A^3 - ABC)E_b + (A^3 - ABC)E_a + (C^2B - A^2C)E_aE_b\right]$$

$$= \frac{1}{D^2}\left[B^2C - A^2B - ABC\,E_b + A^3 E_b + A^3 E_a + C^2 B\,E_aE_b - A^2 C\,E_aE_b\right]$$

$$= \frac{1}{D^2}\Big(B(BC-A^2) - A E_b (BC-A^2) - A E_a (BC-A^2) + E_aE_b\,C\,(BC-A^2)\Big]$$

$$= \frac{1}{D}\left[B - AE_b - AE_a + E_aE_b C\right]$$

$$= \frac{C}{D}\left[\frac{B}{C} - \frac{AE_b}{C} - A\frac{E_a}{C} + E_aE_b + \frac{A^2}{C^2} - \frac{A^2}{C^2}\right]$$

$$= \frac{C}{D}\left(E_aE_b - \frac{A}{C}E_a - \frac{A}{C}E_b + \frac{A^2}{C^2} + \underbrace{\frac{B}{C} - \frac{A^2}{C^2}}_{\frac{BC-A^2}{C^2}}\right)$$

$$= \frac{C}{D}\left(E_a - \frac{A}{C}\right)\left(E_b - \frac{A}{C}\right) + \frac{1}{C}.$$