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Statistics C183/C283 **Instructor: Nicolas Christou**

Implied volatilities

One of the most important uses of the Black-Scholes model is the calculation of implied volatilities. These are the volatilities implied by the option prices observed in the market. Given the price of a call option, the implied volatility can be computed from the Black-Scholes formula. However σ cannot be expressed as a function of S_0, E, r, t, c and therefore a numerical method must be employed:

a. By trial and error. Begin with some value of σ and compute c using the Black-Scholes model. If the price of c is too low (compare to the market price) increase σ and iterate the procedure until the value of c in the market is found. Note: the price of the call increases with volatility.

b. Use the method of Newton-Raphson to estimate σ. The method works as follows:

$$c = S_0\Phi(d_1) - \frac{E}{e^{rt}}\Phi(d_2) \Rightarrow f(\sigma) = S_0\Phi(d_1) - \frac{E}{e^{rt}}\Phi(d_2) - c = 0.$$

$$d_1 = \frac{ln(\frac{S_0}{E}) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

$$d_2 = \frac{ln(\frac{S_0}{E}) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$

To find σ we begin with an initial value σ_0 and iterate as follows:

$$\sigma_{i+1} = \sigma_i - \frac{f(\sigma_i)}{f'(\sigma_i)}$$

$$i = 0$$

$$\sigma_1 = \sigma_0 - \frac{f(\sigma_0)}{f'(\sigma_0)}$$

$$i = 1$$

$$\sigma_2 = \sigma_1 - \frac{f(\sigma_1)}{f'(\sigma_1)}$$

$$\vdots$$

The procedure stops when the $|\sigma_{n+1} - \sigma_n|$ is small.

Note:
The derivative of $f(\sigma)$ above is

$$f'(\sigma) = S_0 f(d_1) \times d_1' - \frac{E}{e^{rt}} f(d_2) \times d_2'$$

where $f(d_1)$ is the density of the standard normal distribution at d_1, i.e.

$$f(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{d_1 - 0}{1})^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}$$

Similarly,

$$f(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{d_2 - 0}{1})^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2}$$

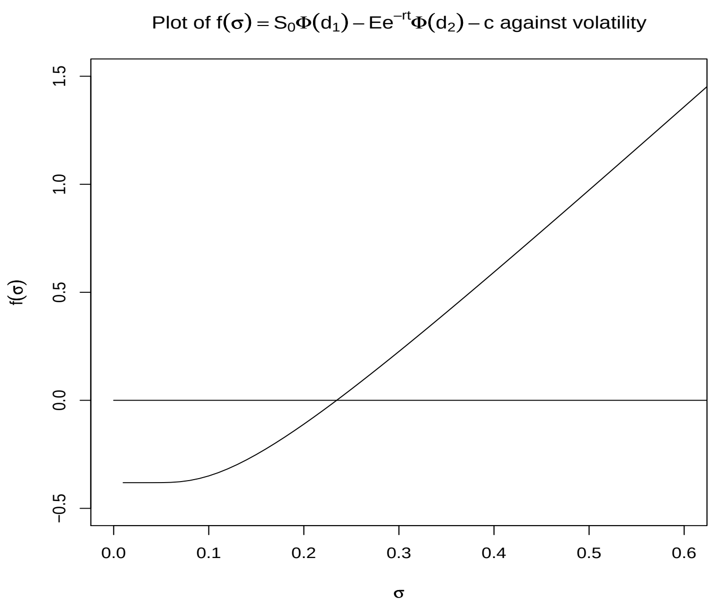
Example:
Suppose the value of a European call is $C = 1.875$ when $s0 = 21, E = 20, r = 0.1, t = 0.25$. Use the method of Newton-Raphson to compute the implied volatility:

```
#Inputs:
s0 <- 21
E <- 20
r <- 0.1
t <- 0.25
c <- 1.875

#Initial value of volatility:
sigma <- 0.10
sig <- rep(0,10)
sig[1] <- sigma
#Newton-Raphson method:
for(i in 2:100){

d1 <- (log(s0/E)+(r+sigma^2/2)*t)/(sigma*sqrt(t))
d2 <- d1-sigma*sqrt(t)
f <- s0*pnorm(d1)-E*exp(-r*t)*pnorm(d2)-c

#Derivative of d1 w.r.t. sigma:
d11 <- (sigma^2*t*sqrt(t)-(log(s0/E)+(r+sigma^2/2)*t)*sqrt(t))/(sigma^2*t)
#Derivative of d2 w.r.t. sigma:
d22 <- d11-sqrt(t)
#Derivative of f(sigma):
f1 <- s0*dnorm(d1)*d11-E*exp(-r*t)*dnorm(d2)*d22

#Update sigma:
sigma <- sigma - f/f1
sig[i] <- sigma
if(abs(sig[i]-sig[i-1]) < 0.00000001){sig<- sig[1:i]; break}
}
```

Here is the vector that contains the volatility at each step:

```
> sig
[1] 0.1000000 0.3575822 0.2396918 0.2345343 0.2345129 0.2345129
```

The implied volatility is $\sigma = 0.2345$.



The graph shows the plot of the function $f(\sigma)$ against σ. The implied volatility is the value of σ such that $f(\sigma) = 0$.