Short sales not allowed, risk free asset exists
Single index model - Ranking stocks

The calculation of optimal portfolios is simplified by using the single index model to rank securities based on the excess return to beta ratio defined as follows:

\[
\text{Excess return to beta} = \frac{\bar{R}_i - R_f}{\beta_i}.
\]

After stocks are ranked using the above ratio the optimum portfolio consists of investing in all stocks for which the excess return to beta is greater than the cut-off point \(C^*\). This cut-off rate is computed as follows:

\[
C^* = \frac{\sigma^2_m \sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\beta_j}{\sigma^2_{ij}}}{1 + \sigma^2_m \sum_{j=1}^{i} \frac{\beta_j^2}{\sigma^2_{ij}}}.
\]

where

\(\bar{R}_j\) Expected return on stock \(j\).
\(R_f\) Return on a riskless asset.
\(\beta_j\) Change in the rate of return of stock \(j\) associated with a 1% change in the market return.
\(\sigma^2_m\) Variance in the market index.
\(\sigma^2_{ij}\) Variance of the error term. Also known as unsystematic risk.

To find \(C^*\) we compute all \(C_i\)'s using portfolios that consist with the first ranked stock, the first and second ranked stocks, the first, second, and third ranked stock etc. We know we have found the cut-off point \(C^*\) when all stocks used in calculating \(C_i\) satisfy:

\[
\frac{\bar{R}_i - R_f}{\beta_i} > C^*.
\]

Once \(C^*\) is found, we know that the optimum portfolio consists of the first \(i\) stocks which satisfy the above inequality. To find the proportion of funds invested in each of these stocks we use:

\[
z_i = \frac{\beta_i}{\sigma^2_{i}} \left( \frac{\bar{R}_i - R_f}{\beta_i} - C^* \right)
\]

Therefore \(x_i = \frac{z_i}{\sum_{i=1}^{n} z_i}\), where \(n\) is equal to the number of stocks consisting the optimum portfolio. Below an example with 16 stocks is presented. Monthly returns on the stocks listed below were selected from January 1980 to February 2001. Using the single index model we estimate the mean return of each stock. The results of the simple regressions for each stock against the market index (DJIA) are summarized in the next table. Also for these data the expected return of the market index is \(\bar{R}_m = 0.01082\) and its variance is \(\sigma^2_m = 0.00192\). Assume \(R_f = 0.005\)
Similarly $z_2 = 1.445$, $z_3 = 1.510$, $z_4 = 0.953$, $z_5 = 0.662$, $z_6 = 0.279$, $z_7 = 1.055$, and $z_8 = 0.324$. The sum of the $z_i$'s is $\sum_{i=1}^{8} z_i = 8.636$. Therefore $x_1 = \frac{2.409}{8.636} = 0.28$, $x_2 = \frac{1.445}{8.636} = 0.17$, $x_3 = \frac{1.510}{8.636} = 0.17$, $x_4 = \frac{0.953}{8.636} = 0.11$, $x_5 = \frac{0.662}{8.636} = 0.08$, $x_6 = \frac{0.279}{8.636} = 0.03$, $x_7 = \frac{1.055}{8.636} = 0.12$, and $x_8 = \frac{0.324}{8.636} = 0.04$. We conclude that the optimum portfolio consists of 28% Consolidated Edison, 17% Merck, 17% Coca Cola, 11% Johnson & Johnson, 8% Pepsi, 3% Texaco, 12% General Electric, and 4% Ford Motor stocks.